Geometry Labs United: An Invitation

Jayadev Athreya, David Dumas, William Goldman, Sergey Grigorian, Rosemary Guzman, Philipp Hieronymi, Sean Lawton, Anton Lukyanenko, Jeremy Tyson, and Aaron Wilson

The purpose of a Geometry Lab is to foster a community to promote research involving all stages of academia, provide mentoring experiences to early career mathematicians, and disseminate mathematics beyond the boundaries of the university. Breakthroughs in science and mathematics often come from experimentation, computation, and communication of technical facts to non-experts. Geometry Labs (also known as Experimental Mathematics Labs) espouse this approach with vertically integrated groups of students and researchers advancing mathematics through experimentation, visualization, and public engagement.

The research community at the core of an experimental mathematics lab produces a synergistic experience: the undergraduates' research resonates with their coursework; graduate students' research blends with mentoring; and faculty foster a deeper connection between research and teaching. In parallel, this shared experience informs public engagement activities that bridge the gap between science and society, paving the way for the next generation of researchers.

Figure 1: The ten member labs of the Geometry Labs United (GLU) network.

Jayadev Athreya is associate professor at the University of Washington. His email address is jathreya@uw.edu.

David Dumas is professor in the Department of Mathematics, Statistics, and Computer Science at the University of Illinois at Chicago. His email address is david@dumas.io.

Philipp Hieronymi is associate professor at the University of Illinois at Urbana–Champaign. His email address is phierony@illinois.edu.

William Goldman is professor at the University of Maryland, College Park. His email address is wmg@math.umd.edu.

Sergey Grigorian is assistant professor at the School of Mathematical and Statistical Sciences, University of Texas Rio Grande Valley. His email address is sergey.grigorian@utrgv.edu.

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Rosemary Guzman is a postdoctoral fellow in the Department of Mathematics at the University of Chicago. Her email address is guzman@math.uchicago.edu.

Sean Lawton is associate professor at George Mason University. His email address is slawton3@gmu.edu.

Anton Lukyanenko is assistant professor at George Mason University. His email address is alukyane@gmu.edu.

Jeremy Tyson is professor and mathematics department chair at the University of Illinois at Urbana–Champaign. His email address is tyson@illinois.edu.

Aaron Wilson is assistant professor at the School of Mathematical and Statistical Sciences, University of Texas Rio Grande Valley. His email address is aaron.wilson@utrgv.edu.

Through this community and shared experience, geometry labs make research and geometry, broadly understood, more accessible. For all students, and especially for those under-represented in STEM, geometry labs serve as an important bridge toward higher mathematics. The emphasis on collaboration, active and experiential research, and outreach make geometry labs both a microcosm of the larger mathematical research community and an example of what it can be. To showcase this diversity, we highlight some of the students and projects, focusing on recent developments in GLU's 18-year history.

Samuel J. Cavazos, Experimental Algebra and Geometry Lab

Figure 2: Samuel J. Cavazos found a formula for the number of representations of free groups over finite fields.

Sam Cavazos (Figure 2) was a member of the Experimental Algebra & Geometry Lab (EAGL) at the University of Texas Rio Grande Valley (UTRGV) from the spring of 2011 to the summer of 2013. During Sam's first year in college he began working with Sean Lawton in EAGL. He was given the task of counting representations of free groups over finite fields (Figure 3). He quickly learned to write code in the computer program Mathematica and conjectured the correct formula by analyzing patterns generated by his program. In his second year, Sam began the project which led to his publication "E-polynomial of SL(2,$\mathbb{C}$)-character varieties of free groups," *Internat. J. Math.* 25 (2014). This result gives an exact counting polynomial over all finite fields and for all free groups of the conjugation orbits of the aforementioned space of representations, as well as the point count of the associated quotient in the category of algebraic varieties. Such spaces are called *character varieties*, central objects in differential geometry and mathematical physics.

Based on his work in EAGL, Sam won a National Science Foundation Graduate Fellowship in 2013. He completed his undergraduate studies in just three years and graduated Summa Cum Laude with a BS in Pure Mathematics and a minor in Philosophy. Sam went on to Northwestern University, earned a Master's degree in mathematics in 2015, and is now a lead data analyst in Chicago.

Sam's research at UTRGV was supported four times by the Louis Stokes Alliances for Minority Participation (LSAMP) Program. He presented his work at the LSAMP conference, the Young Mathematicians Conference, the Society for Advancement of Chicanos and Native Americans in Science (SACNAS) conference, the Hispanic, Engineering, Science, and Technology (HESTEC) Week, and numerous faculty and student research seminars at UTRGV. He won the top research prizes for his poster and presentations at HESTEC and SACNAS (receiving national attention). Additionally, Sam gave mathematical community outreach presentations to middle school students and teachers through EAGL's outreach endeavors.

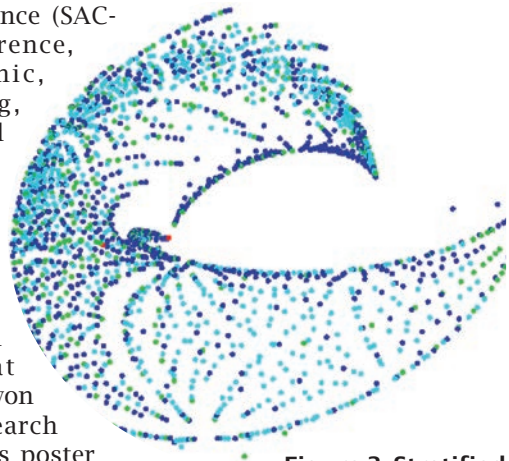


Figure 3: Stratified Finite Field Points on SL_2.

According to Sam, "The lessons I learned during my time at the Experimental Algebra and Geometry Lab extend far beyond the world of mathematics. It taught me how to solve problems that make a positive impact on my community."

Kristine Hampton, Washington Experimental Mathematics Lab (WXML)

Figure 4: Kristine Hampton studied the statistical properties of the discrete logarithm viewed as a permutation.

Kristine Hampton (Figure 4) joined the WXML at the University of Washington in its first quarter of existence, Spring 2016. She worked alongside Matthew Ogi and Qiang Yu on a project co-mentored by Jayadev Athreya, Chris Hoffman, Jacob Richey, and Dan Bragg, studying the statistical properties of the discrete logarithm viewed as a permutation. This project combined number theory, group theory, combinatorics, probability, and programming. The project developed into a study of the actions of linear groups over finite fields on projective spaces, motivated by questions of V.I. Arnold. Kristine led the team through fascinating numerical and theoretical questions, particularly analyzing the number of cycles and number of fixed points in the discrete logarithm, and comparing them to those of

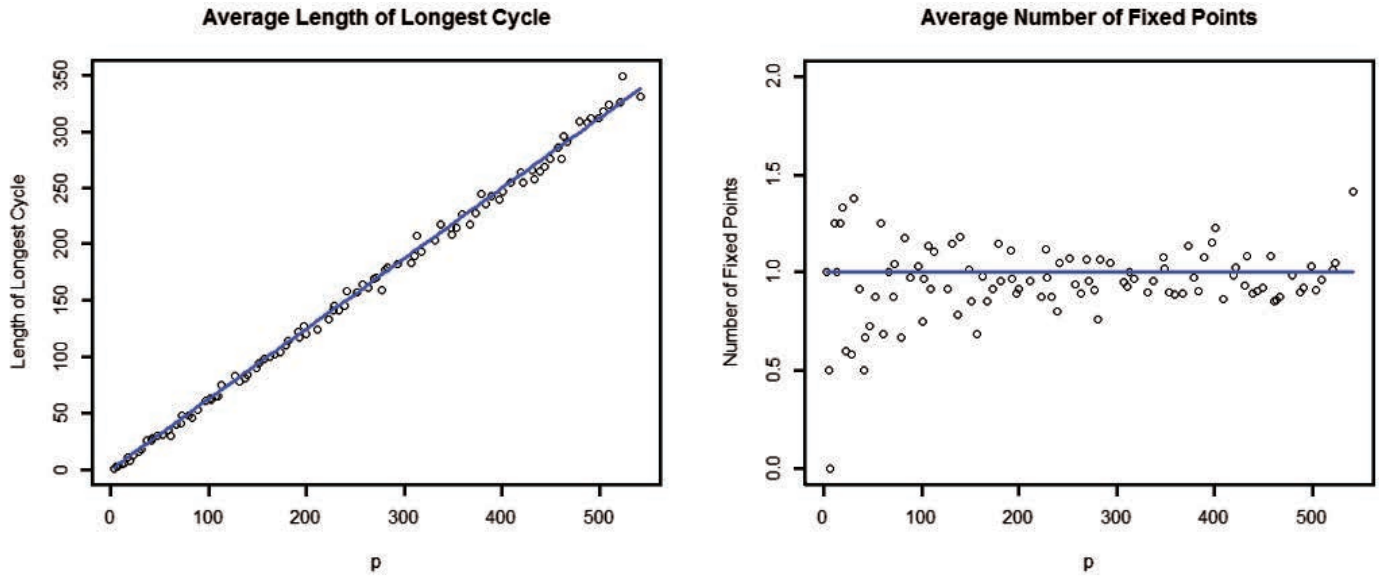


Figure 5: Asymptotics of number of cycles and length of longest cycles in discrete logarithms.

randomly chosen permutations (Figure 5). After working with the WXML through the 2016–17 academic year, Kristine joined UW as a PhD student in Autumn 2017, where she is planning on studying algebraic geometry.

In addition to her research work, Kristine has helped start and develop the WXML and "Math for Love" math teacher circles, a monthly math circle for elementary and middle school math teachers. These math circles, managed by Christine, are a collaboration between "Math for Love," a Seattle-based mathematics organization, and the WXML. Kristine was instrumental in securing funding from the American Institute of Mathematics and also getting the participating teachers professional development credits. She continues to successfully manage the math circles as a PhD student.

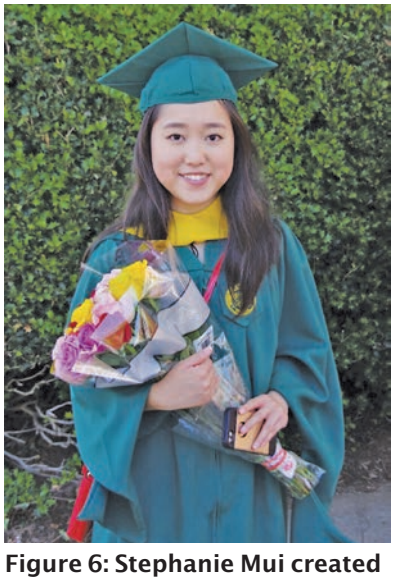


Figure 6: Stephanie Mui created sinusoidal embeddings of flat tori and Nash spheres in 3D.

Stephanie Mui, Mason Experimental Geometry Lab

Stephanie Mui (Figure 6) worked with Sean Lawton at the Mason Experimental Geometry Lab (MEGL) at George Mason University from the summer of 2015 to the spring of 2017. Stephanie's work visualizing geometric surfaces in 3D (Figure 7) qualified her for the 2016 Intel International Science and Engineering Fair (Intel ISEF), where she won the first place award of US$2,000 from the American Mathematical Society.

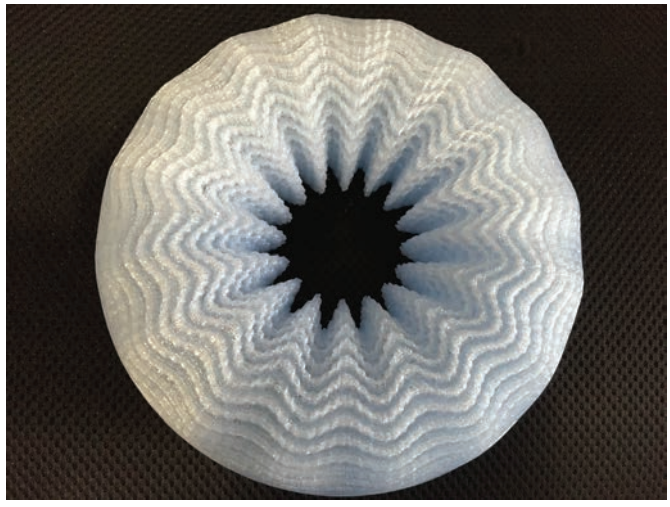


Figure 7: A 3D print of a flat torus.

Her thesis research concerned visualizing embedded geometric manifolds. She used computer programming and 3D printing to experimentally explore her ideas as she studied relevant theory and developed new ideas. See `meglab.wikidot.com/visualization` for examples of her 3D prints and stereographic 3D movies.

When Stephanie was not working on research or earning straight A's, she assisted with MEGL's community outreach, inspiring young people in the community to appreciate mathematics. She also gave many research presentations including a talk at the Undergraduate Mathematics Symposium at the University of Illinois at Chicago.

Stephanie graduated *Summa Cum Laude*, earning both a BS in Pure Mathematics with honors and an MS in Mathematics. After graduating Stephanie went to work on her PhD at the Courant Institute of Mathematical Sciences at NYU with a full research fellowship. Stephanie was the

youngest graduate in GMU's history at age 17; in fact, she earned her MS and BS before her high school diploma.

According to Stephanie, "MEGL is the prime reason I am continuing my studies in math today. My mentor at MEGL sparked my interest in differential geometry and provided me the rigorous training and insightful roadmap to conduct research."

Matthew Staples, Washington Experimental Mathematics Lab

Matthew Staples (Figure 8) joined the WXML at the University of Washington in its first quarter of existence, Spring 2016, working alongside Wenbo Gao and Maria Osborne on a project mentored by Jayadev Athreya and Dia Taha. Matt's group worked to understand the properties of the new pentagonal tiling of the plane recently discovered by Mann, McLoud, and Von Derau at the University of Washington Bothell (also as part of an undergraduate research project). They studied statistical properties of this tiling and the billiard flow in this rational pentagon, and showed that the associated surface had a large group of affine symmetries.

Figure 8: Matthew Staples studied properties of a new pentagonal tiling of the plane showing that the associated surface has a large group of symmetries.

In addition to his WXML research work, Matt helped develop WXML's website and worked on several 3D printing and fabrication projects. Matt graduated from UW in summer of 2016, and is currently an associate data warehouse analyst at the Comtech Telecommunications Corporation in Seattle.

According to Matt, "Being a part of WXML was the best part of my university experience."

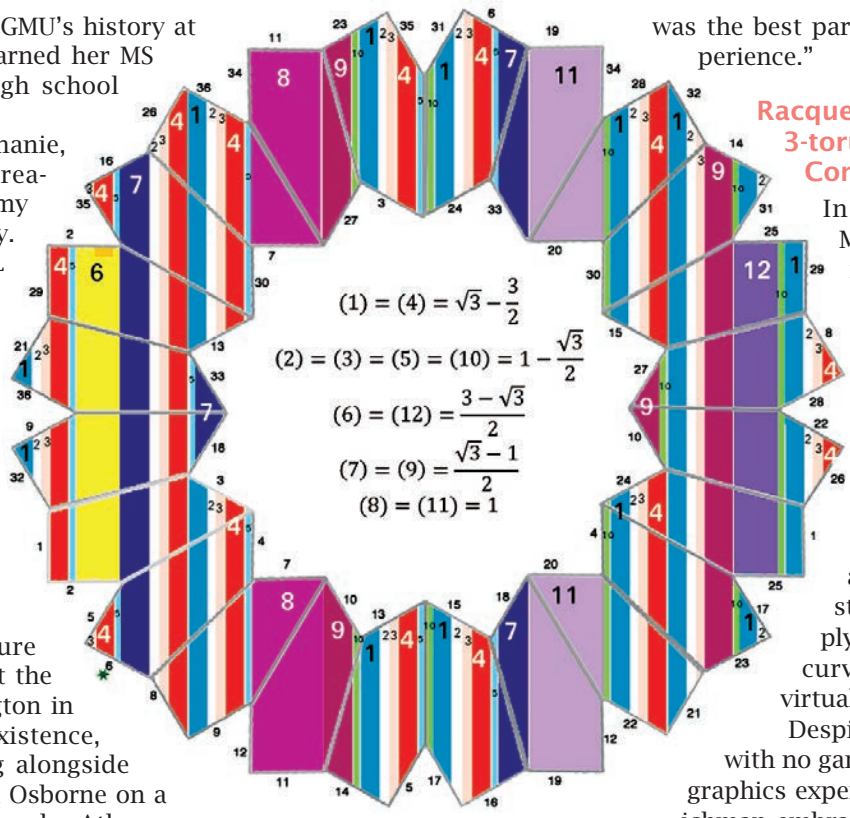


Figure 9: A cylinder decomposition of the translation surface associated to billiards in the Bothell pentagon.

Racquetball in the 3-torus, Mathematical Computing Laboratory

In this project at the Mathematical Computing Lab at the University of Illinois at Chicago, undergraduate researchers Horalia Armas and Brandon Reichman worked with graduate mentor Hai Tran and faculty supervisor David Dumas (Figure 10) on a project to demonstrate features of multiply-connected spaces and curved geometries using virtual reality.

Despite starting the project with no game development or 3D graphics experience, Armas and Reichman embraced the challenge and had considerable success. Using the Unity 3D engine and its associated integrated development environment, they created a first-person VR racquetball game that can be played in the 3-dimensional torus and several other Euclidean orbifold geometries.

Notably, this racquetball game is designed for a single player; the ball is passed back and forth along closed geodesics in the ambient space. By taking advantage of the

Figure 10: Faculty supervisor David Dumas, undergraduate Brandon Reichman, and graduate mentor Hai Tran, along with undergraduate Horalia Armas (not pictured here) developed a virtual reality ManifoldBall to understand multiply connected spaces and curved geometries.

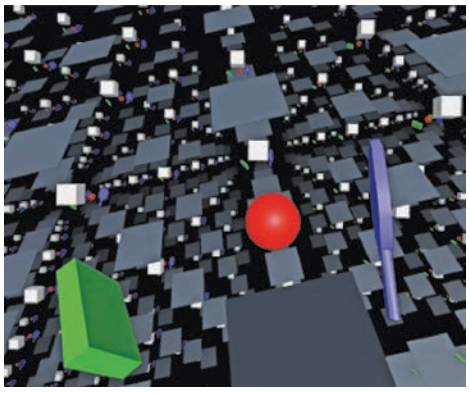


Figure 11: In-game screenshot of *ManifoldBall*.

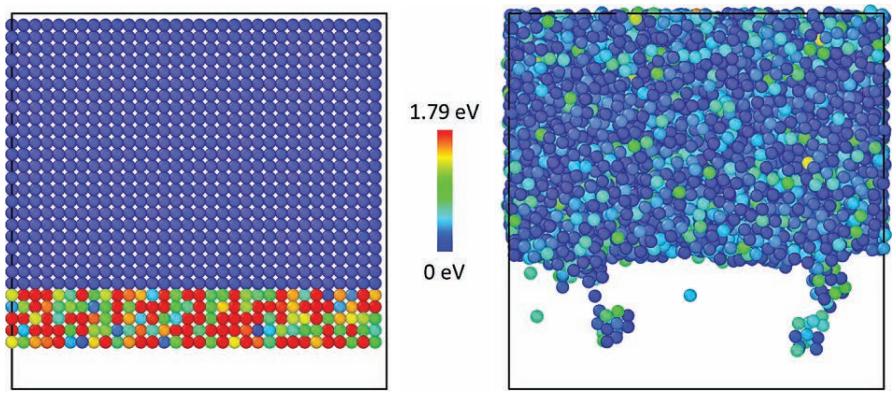


Figure 13: Model of melting Gallium.

player's expectations about the basic goal of the game (i.e. to hit the ball), the game embeds exploration of the geometry of the 3-torus in a familiar and recreational setting.

The resulting game, *ManifoldBall*, and the group's project were featured in a March 2017 article in the journal *Nature*.[1]

To play the game, visit `mcl.math.uic.edu/spring-2017-projects`.

Statistical Quantum Mechanics for Graphs and CW-Complexes, Illinois Geometry Lab

In this project at the Illinois Geometry Lab (IGL) at the University of Illinois at Urbana-Champaign, the research team led by faculty mentor Ivan Contreras and graduate team leader Sarah Loeb (now a postdoc at the College of William & Mary) developed a model for quantum mechanics and

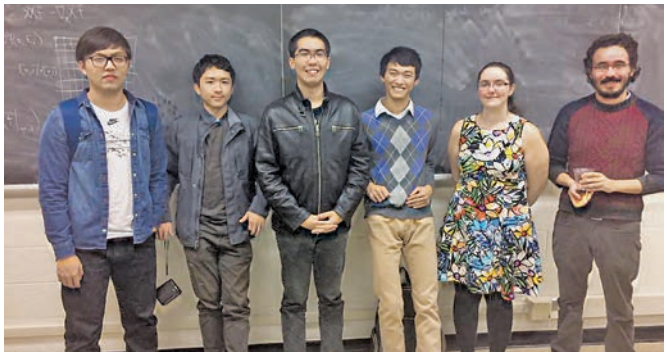


Figure 12: The research team Statistical Quantum Mechanics for Graphs and CW-Complexes developed a machine-learning based algorithm to apply graph quantum mechanics to text analysis.

thermodynamics. In this model, the physical system is confined to a finite graph or a CW-complex, and the evolution of the system is prescribed by a discretized version of the Schrödinger or heat equation, respectively. The team used this model to better understand the topology of graphs and CW-complexes, as well as to provide a combinatorial meaning to the partition function of quantum mechanics and thermodynamics (Figure 13). As an application, they developed a machine-learning based algorithm to apply graph quantum mechanics to text analysis.

Three papers resulted from this research, all including undergraduates as co-authors. Undergraduate students in the team have presented their work at various regional and national conferences. In particular, Rodrigo Araiza presented at the Latinx Conference, and Chengzheng Yu at the 2018 Joint Math Meetings. In addition, undergraduate Michael Toriyama won a prestigious Barry Goldwater Scholarship, and his fellow team member Boyan Xu is starting graduate studies in mathematics at UC Berkeley in Fall 2018. This project exemplifies the IGL's mission to make substantial contributions to mathematical knowledge, while simultaneously fostering the professional development of both undergraduate and graduate students.

How Does a Lab Take Shape?

The narratives above demonstrate the vast diversity of Experimental Mathematics Labs: their projects, participants, and experiences. Likewise, the labs themselves take many shapes, and each one develops based on the resources, needs, and interests of its department. The goal of a founding director is to leverage these components to build a vertically-integrated research community focused on visualization, computation, and public engagement. The particular way in which this happens varies from lab to lab.

Ultimately, an Experimental Mathematics Lab is an organization that facilitates vertically integrated projects within a mathematics department. An approach used by most labs is to run term-long projects during the academic year. These projects are led by a faculty member, overseen if possible by a graduate student, and implemented by a team of 2–4 undergraduate students earning course credit or financial compensation.

Ideally, everyone is engaged at the level and time commitment appropriate to their experience: the undergraduate students work up to 10 hours per week and contribute computations and technology expertise; the graduate student guides the team, exploring and explaining relevant theory and meeting with the undergraduates for one hour a week; the faculty mentor oversees the experience,

[1] *Mathematicians create warped worlds in virtual reality,* Nature *543, 473 (23 March 2017).* `https://www.nature.com/news/mathematicians-create-warped-worlds-in-virtual-reality-1.21689`.

suggesting the initial problem and regularly joining the weekly meetings.

The lab is run by a faculty director, with further assistance from faculty members, graduate students, or undergraduate students according to the size of the lab. To prepare for each new term, lab leadership solicits projects from faculty, advertises to the students, and forms collaborative teams. During the semester, consultation is provided to teams, and all-lab meetings are arranged to showcase progress to the lab's members, culminating in a broadly-advertised public poster session or a research symposium.

In parallel, the lab leadership (often involving a dedicated outreach manager) develops and coordinates community partnerships and engagement activities, providing a bridge between university mathematics and the broader public. These activities are implemented by the outreach manager, as well as volunteer lab members at all levels. The community engagement has two important effects: the university students leading the activities gain confidence in their own ability to do and present mathematics, and the K–12 activity participants show improved attitudes towards mathematics through the near-peer effect.[2]

To build up to this level of infrastructure, a founding director must ignite interest within their department. Initial investment by the department and student body can be created by showcasing projects from other labs and advertising the use of cutting-edge technology in research (such as 3D printing, virtual reality, and parallel computing). Once this advertising creates a seed community of interested parties, projects can be started, and, if possible, a space can be established for joint meetings, presentations, and collaborative student work. While such a space can be hard to obtain, and strictly speaking is not necessary, it is useful in creating a cohesive, vertically integrated community that facilitates sharing of expertise across projects.

Call for More Labs

If you're interested in developing your own experimental mathematics lab, GLU can help. We encourage you to join us at our special session at the 2019 Joint Mathematics Meetings in Baltimore (January, 2019) or the bi-annual GLU Conference (Summer 2019) at George Mason University; visit us at `geometrylabs.net` (where you can find a lab starter kit, created by Dia Taha, Leif Svensson, Jonah Ostroff, and the WXML leadership team, building on materials created at the IGL by Noel DeJarnette, Anton Lukyanenko, and Grace Work), or email us at `glu@geometrylabs.net`.

[2] *A. Wilson & S. Grigorian, The near-peer mathematical mentoring cycle: studying the impact of outreach on high school student attitudes toward mathematics,* Int. J. Math. Ed. Sci. Tech. *49 (2018).*

ABOUT THE AUTHORS

Jayadev Athreya

Jayadev Athreya co-founded the Illinois Geometry Lab, and directed it from 2011 to 2015. In 2012, he co-founded Geometry Labs United. He founded the Washington Experimental Mathematics Lab in 2015, and has directed it since. He sees experimental mathematics labs as vehicles for advancing equity and diversity in mathematics.

David Dumas

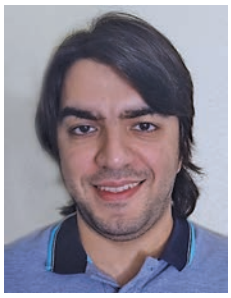
David Dumas is a co-founder of the Mathematical Computing Laboratory at University of Illinois at Chicago.

Philipp Hieronymi

Philipp Hieronymi is the director of the Illinois Geometry Lab.

William Goldman

William Goldman directs the Experimental Geometry Lab. He studied with Bill Thurston, Dennis Sullivan, and Morris Hirsch at Princeton and Berkeley in the 1970s. In 2000 he co-founded the EGL, where he mentored Sean Lawton and Anton Lukyanenko. In 2012, after three geometry labs had been formed, he co-founded Geometry Labs United. He co-directed a summer 2018 REU at ICERM at Brown University.



Sergey Grigorian

Sergey Grigorian is currently a co-director of the Experimental Algebra and Geometry Lab. He is working to integrate geometry lab activities with the university curriculum within a community engaged scholarship framework.

Anton Lukyanenko is a past member of the Experimental Geometry Lab (as an undergraduate student), a co-founder of the Illinois Geometry Lab (as a graduate student) and the Laboratory of Geometry at Michigan (as a postdoc), and an assistant director of the Mason Experimental Geometry Lab (as a faculty member). In 2012, he co-founded Geometry Labs United.

Anton Lukyanenko

Rosemary Guzman

Rosemary Guzman served as assistant director of the Illinois Geometry Lab from 2015 to 2017.

Jeremy Tyson served as director of the Illinois Geometry Lab from 2015 to 2017.

Jeremy Tyson

Sean Lawton

Sean Lawton was the founding director of the Experimental Algebra & Geometry Lab in Texas from 2009 to 2014. In 2012, he co-founded Geometry Labs United. In 2014 he founded the Mason Experimental Geometry Lab and has since directed it. He studied with William Goldman at the Experimental Geometry Lab in Maryland as a student and post-doc.

Aaron Wilson is one of the current directors of the Experimental Algebra and Geometry Lab. He is interested in measuring the positive changes observed in students involved with the lab.

Aaron Wilson

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