PRELIMINARY EXAM IN GEOMETRY AND TOPOLOGY
SEPTEMBER 2023

Although the exam is in three parts, you may use results from all three quarters in answering these questions. You should answer **three** problems from each part. If you attempt more than three problems from one part, the three problems with highest scores will count.

Full credit requires proving that your answer is correct. You may quote and use theorems and formulas. But if a problem asks you to state or prove a theorem or a formula, you are expected to provide full details.

Write the last four digits of your student ID at the top of each page, together with the page number, and the total number of pages at the top of the first page. Double space your answers for readability.

Part I

Answer **three** of the following five problems.

1) Construct a vector field on the 3-sphere with no fixed points and compute its flow.
2) Let ω be a exact two-form. Show that ω^n is exact for all $n > 0$.
3) State the definition of the injectivity radius of a Riemannian manifold. Prove that if M is a compact Riemannian manifold, there is $\epsilon > 0$ such that the injectivity radius at each point is greater than ϵ.
4) Construct a **flat** Riemannian metric g on the plane $\mathbb{R}^2$ which is not constant, i.e. at least one of the three coefficients $g(\partial_x, \partial_x)$, $g(\partial_x, \partial_y)$, $g(\partial_y, \partial_y)$, is not a constant.
5) Let (M, g) be a Riemannian manifold, let Ω be the Riemannian volume form, and let ∇ be the Levi–Civita connection. Prove that $\nabla\Omega = 0$.

Part II

Answer **three** of the following five problems.

1) Let M be a compact manifold (possibly with boundary) of dimension n. Let $B \subset M$ be the interior of an embedded closed n-dimensional ball, and let M' be the complement of B. State and **prove** a formula for the Euler characteristic of M' in terms of the Euler characteristic of M.

2) Show that the fundamental group of a topological group is abelian.

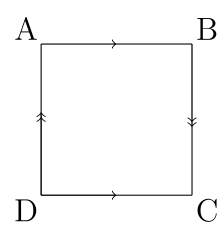
3) Let G be a compact Lie group with $\dim(G) > 0$. Show that the Euler characteristic of G is zero.

4) Let M be a manifold. Let $C_*(M)$ be the complex of singular chains, and let $C_n^\infty(M) \to C_n(M)$ be the subset of singular chains generated by the those simplices $\sigma : \Delta^n \to M$ given by infinitely differentiable maps.

   a) Prove that $C_*^\infty(M)$ is a subcomplex of $C_*(M)$.

   b) Prove that the inclusion $C_*^\infty(M) \hookrightarrow C_*(M)$ is a quasi-isomorphism of complexes (both complexes have isomorphic homology).

5) Let K be the Klein bottle, i.e. the space obtained from the square ABCD by gluing the edges AB and DC, and gluing the edges BC and DA:



   Compute the homology groups $H_i(K; \mathbb{F}_p)$, for all i and p.

Part III

Answer **three** of the following five problems.

1) Prove that the integral cohomology of a compact manifold is finitely generated.
2) Prove that there is no continuous function $f : \mathbb{CP}^2 \to S^2 \times S^2$ that induces an isomorphism on H^4.
3) Find an example of a map $S^a \to S^b$, $a \neq b$, that is not null homotopic. Prove that the map you described is not null homotopic.
4) Prove that the tangent bundle of $S^2 \times S^4$ is not trivial.
5) Let $X = \mathbb{CP}^1 \setminus \{0, 1\}$. Find compactly supported closed one-forms

$$\omega_1, \ldots, \omega_n \in \Omega^1_c(X)$$

whose images in the compactly supported de Rham cohomology group $H^1_c(X, \mathbb{R})$ form a basis.