PRELIMINARY EXAM IN GEOMETRY AND TOPOLOGY
JUNE 2023

Although the exam is in three parts, you may use results from all three quarters in answering these questions. You should answer **three** problems from each part. If you attempt more than three problems from one part, the three problems with highest scores will count.

Full credit requires proving that your answer is correct. You may quote and use theorems and formulas. But if a problem asks you to state or prove a theorem or a formula, you are expected to provide full details.

Write the last four digits of your student ID at the top of each page, together with the page number, and the total number of pages at the top of the first page. Double space your answers for readability.

Part I

Answer **three** of the following five problems.

1) Let $S^7 = \{x \in \mathbb{R}^8 \mid |x|^2 = 1\}$ be the unit sphere in $\mathbb{R}^8$, with the orientation induced from the standard orientation on $\mathbb{R}^8$. Let η and ω be differential 3-forms on $\mathbb{R}^8$. Show that

$$\int_{S^7} d\omega \wedge \eta = \int_{S^7} \omega \wedge d\eta.$$

2) Let $\omega = dx \wedge dy \wedge dz$ be the standard volume form on $\mathbb{R}^3$. Given a vector field X on $\mathbb{R}^3$ of the form

$$X = f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y} + h\frac{\partial}{\partial z},$$

the divergence of X is the function

$$\operatorname{div} X = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}.$$

Show that $\mathcal{L}_X\omega = 0$ (that is, the flow induced by X preserves ω) if and only if X is divergence free (that is, if and only if $\operatorname{div} X = 0$).

3) Let M and N be compact smooth manifolds. Show that $M \times N$ is orientable if and only if both M and N are orientable.

4) Let $M_n(\mathbb{R})$ denote the vector space over $\mathbb{R}$ of $n \times n$ matrices, and for $A \in M_n(\mathbb{R})$, let A^T be the transpose of A.

   Let $S_n(\mathbb{R}) = \{A \in M_n \mid A = A^T\}$ be the vector space over $\mathbb{R}$ of $n \times n$ symmetric matrices. Let

$$\mathrm{O}(n) = \{A \in M_n \mid AA^T = I\}$$

   be the orthogonal group.

   By considering the map $f : M_n(\mathbb{R}) \to S_n(\mathbb{R})$ given by the formula

$$f(A) = AA^T,$$

   or otherwise, show that $\mathrm{O}(n)$ is a manifold, and calculate the dimensions of its components.

5) Let $h\colon [0, \infty) \to (0, \infty)$ be a nowhere vanishing function. Equip $\mathbb{R}^2$ with polar coordinates $(x_1, x_2) = (r, \theta)$. Let g be a Riemannian metric with Christoffel symbols (relative to the coordinates $(x_1, x_2) = (r, \theta)$)

$$\Gamma^1_{ij} = \begin{pmatrix} 0 & 0 \\ 0 & -h'(r) \end{pmatrix}, \qquad \Gamma^2_{ij} = \frac{1}{2\,h(r)} \begin{pmatrix} 0 & h'(r) \\ h'(r) & 0 \end{pmatrix}.$$

   Let r_0 be a critical point for h. Fix $b \in \mathbb{R}$. Show that the parameterized circle

$$\gamma(t) = (r_0, bt),$$

   centered at the origin, is a geodesic.

Part II

Answer **three** of the following five problems.

1) Let X be the union of the unit sphere in $\mathbb{R}^3$ and the line connecting the north and south poles. What is the fundamental group of this topological space?
2) Show that the fundamental group of a topological group is abelian.
3) If X and Y are pointed topological spaces, $X \vee Y$ is the space obtained by taking the disjoint union of X and Y and identifying the basepoints. Prove that $S^2 \vee S^4$ is not homotopy equivalent to $\mathbb{C}P^2$.
4) Given a natural number $n > 0$, find a topological space X such that the abelian group $H_n(X)$ has non-zero torsion elements.
5) Let $f : S^n \to S^n$ be the antipodal map $f(x) = -x$. What is the degree of f?

Part III

Answer **three** of the following five problems.

1) Suppose that M is a manifold and G is a finite group acting freely on M (that is, every non-identity element of g acts without fixed points). Let $H^i_{\mathrm{dR}}(M)^G$ be the subspace of G-invariants in $H^i_{\mathrm{dR}}(M)$. Prove that $H^i_{\mathrm{dR}}(M/G) = H^i_{\mathrm{dR}}(M)^G$.
2) Compute the de Rham cohomology ring of $S^n \times \mathbb{C}P^n$. You may quote the Betti numbers of S^n and $\mathbb{C}P^n$ without proof.
3) Let M be a 6-dimensional compact oriented manifold. Show that $\dim H^3(M;\mathbb{R})$ is even.
4) Let (M, g) be a compact Riemannian manifold. Suppose that M can be covered by k balls of radius 1 and that these balls are geodesically convex. Show that $\dim H^i(M;\mathbb{R}) \leq \binom{k}{i}$.
5) Let $X = \{(x, y) \in S^2 \times S^2 \subset \mathbb{R}^2 \times \mathbb{R}^2 \mid x \perp y\}$. Compute the cohomology groups $H^i(X;\mathbb{Z})$.