PRELIMINARY EXAM IN ANALYSIS
SEPTEMBER 2023

INSTRUCTIONS:

(1) This exam has **three** parts: I (measure theory), II (functional analysis), and III (complex analysis). Do **three** problems from each part. If you attempt more than three problems in one part, then the three problems with highest scores will count.

(2) In each problem, full credit requires proving that your answer is correct. You may quote and use theorems and formulas. But if a problem asks you to state or prove a theorem or a formula, you need to provide the full details.

Part I. Measure Theory

Do **three** of the following five problems.

(1) (a) State Fatou's Lemma.
 (b) Use the Monotone Convergence Theorem to prove Fatou's Lemma.
 (c) Show that the inequality in Fatou's Lemma can be strict.

(2) Let $(X, \mathcal{F}, \mu)$ be a measure space and $\{f, f_n, n \geq 1\}$ a sequence of nonnegative integrable functions. Consider the following statements:
 (a) $f_n \to f$ almost everywhere with respect to μ and $\int f_n d\mu \to \int f d\mu$.
 (b) $\int |f_n - f| d\mu \to 0$.
 (c) $f_n \to f$ in measure and $\int f_n d\mu \to \int f d\mu$.
 For each of the six ordered pairs of the above statements, either prove that the first one implies the second one or find a counterexample showing that the first one does not always imply the second one.

(3) Let $(X, \mathcal{F}, \mu)$ be a finite measure space. Let f be a bounded function with $\mu(\{|f| > 0\}) > 0$. Prove that the ratio
$$\frac{\int_X |f|^{n+1} d\mu}{\int_X |f|^n d\mu}$$
converges as $n \to \infty$ and calculate its limit.

(4) (a) State the Lebesgue-Radon-Nikodym Theorem.
 (b) Suppose μ, ν are probability measures such that $\nu \ll \mu$ and let $\lambda = \mu + \nu$. Show that if $f = \frac{d\nu}{d\lambda}$ then $0 \leq f < 1$, μ-a.e. and justify that $\frac{d\nu}{d\mu} = \frac{f}{1-f}$.

(5) Let $f \in L^1(\mathbb{R})$ be a Lebesgue integrable function. Suppose that $\int_I f d\lambda = 0$ for every interval I with Lebesgue measure $\lambda(I) = 2$. Show that $f = 0$ almost everywhere. Is the condition $f \in L^1(\mathbb{R})$ necessary?

Part II. Functional Analysis

Do **three** of the following five problems.

(1) Let X be a Banach space with the norm $\|\cdot\|$.
   (a) Define the weak convergence of a sequence $\{x_n\}$ in X to an element $x \in X$.
   (b) Show that the norm function $x \mapsto \|x\|$ is lower semicontinuous under the weak convergence.

(2) Let c_0 be the space of real sequences $x = \{x_n, n \in \mathbb{N}\}$ such that the limit $\lim_{n\to\infty} x_n$ exists and is finite. Define the norm $\|x\| = \sup_{n\in\mathbb{N}} |x_n|$. Show that c_0 is a Banach space under this norm and identify explicitly its dual space c_0^*.

(3) Suppose $I = [0,1]$ and $0 < \alpha < 1$. Define

$$\|f\|_\alpha = \sup_{x\in I} |f(x)| + \sup_{x,y\in I} \frac{|f(x) - f(y)|}{|x-y|^\alpha}.$$

Let $C^{0,\alpha}(I)$ be the space of continuous functions f defined on I such that $\|f\|_\alpha < \infty$.
   (a) Show that $C^{0,\alpha}(I)$ is a Banach space under the norm $\|\cdot\|_\alpha$.
   (b) Show that $C^{0,\beta}(I) \subset C^{0,\alpha}(I)$ for $0 < \alpha \leq \beta < 1$.
   (c) Show that the embedding $i_{\alpha,\beta} : C^{0,\beta}(I) \to C^{0,\alpha}(I)$ is compact for $0 < \alpha < \beta < 1$.

(4) Suppose that $f \in \mathscr{S}(\mathbb{R})$ is a Schwartz function and its Fourier transform $\hat{f}$ never vanishes. Show that the span of its translates $\{f_h, h \in \mathbb{R}\}$ is dense in $L^2(\mathbb{R})$. Here $f_h(x) = f(x+h)$.

(5) The principal value function $P = \text{p.v.}\left(\frac{1}{x}\right)$ is defined by

$$Pf = \text{ p.v.} \int_{\mathbb{R}} \frac{f(x)}{x}\, dx = \lim_{\epsilon\downarrow 0} \int_{\{|x|\geq\epsilon\}} \frac{f(x)}{x}\, dx, \qquad f \in \mathscr{S}(\mathbb{R}).$$

   (a) Show that P is well defined on the space of Schwartz functions $\mathscr{S}(\mathbb{R})$.
   (b) Show that P is a tempered distribution on $\mathbb{R}$ and find its Fourier transform.

Part III. Complex Analysis

Do **three** of the following five problems.

(1) Let $f : \Omega \to \mathbb{C}$ be analytic, where Ω contains the closed unit disk $\overline{D(0,1)}$. Let Γ be the image of the boundary $\partial D(0,1)$ under f, i.e., Γ is parametrized by $\theta \mapsto f(e^{i\theta})$, $\theta \in [0, 2\pi]$. Show that

$$\text{length}(\Gamma) := \int_\Gamma |dz| \geq 2\pi |f'(0)|.$$

(2) (a) Find a holomorphic bijection from the half infinite strip

$$S = \{z \in \mathbb{C} : \operatorname{Re} z > 0, \quad \operatorname{Im} z \in (0,1)\}$$

to the upper half plane $\mathbb{H}$.

(b) Suppose that f is an entire function such that $f(0) = 3 + 4i$ and $|f(z)| \leq 5$ for all $z \in D(0,1)$. What are the possible values of $f'(0)$?

(3) (a) Let $f, g : \mathbb{C} \to \mathbb{C}$ be two entire functions, such that $|f(z)| \leq |g(z)|$ for all z. Show that there is a constant $c \in \mathbb{C}$ such that $f(z) = cg(z)$ for all z.

(b) Find all entire functions f which satisfy $|f'(z)| \leq |f(z)|$ for all z.

(4) (a) Let $f : \Omega \to \mathbb{C}$ be analytic, and let $a \in \Omega$. Show that there is an integer $k \geq 0$, and an analytic function $h : \Omega \to \mathbb{C}$ satisfying $h(a) \neq 0$, such that

$$f(z) = (z-a)^k h(z), \text{ for all } z \in \Omega.$$

(b) Let $f : \Omega \to \mathbb{C}$ be a function (not necessarily continuous). Suppose that f^2 and f^3 are analytic on Ω. Show that f is analytic.

(5) Use contour integration to compute the integral

$$\int_0^\infty \frac{\sin(x)}{x(x^2+1)}\, dx.$$