PRELIMINARY EXAM IN ANALYSIS JUNE 2023

INSTRUCTIONS:

(1) This exam has **three** parts: I (measure theory), II (functional analysis), and III (complex analysis). Do **three** problems from each part. If you attempt more than three problems in one part, then the three problems with highest scores will count.

(2) In each problem, full credit requires proving that your answer is correct. You may quote and use theorems and formulas. But if a problem asks you to state or prove a theorem or a formula, you need to provide the full details.

Part I. Measure Theory

Do **three** of the following five problems.

(1) (a) State the Dominated Convergence Theorem.
(b) Let μ be a probability measure on $[0, \pi/2]$ with $\int x d\mu = \pi/4$. Show that

$$\sum_{n=1}^{\infty} \int (1 - \sqrt{\sin x})^n x \sqrt{\sin x}\, d\mu$$

is finite and compute its value.

(2) Let $p, q > 1$ with $1/p + 1/q = 1$. Given $f \in L_p([0,\infty))$ nonnegative, let

$$F(x) = \frac{1}{x} \int_0^x f(y) dy.$$

Show that $F \in L_p([0,\infty))$ and

$$q = \inf \left\{ \alpha > 0 : \|F\|_p \leq \alpha \|f\|_p, \forall f \in L_p([0,\infty)) \text{ nonnegative} \right\}.$$

(3) Let μ be a finite measure on a measurable space $(X, \mathcal{F})$. For f, g measurable, set

$$d(f,g) = \int_X \min\{|f-g|, 1\} d\mu.$$

Show that d is a metric and that a sequence of measurable functions f_n converges in measure to f if and only if $d(f_n, f) \to 0$.

(4) Let $A \subset \mathbb{R}$ be a subset that has a positive Lebesgue measure $\lambda(A) > 0$. Show that A contains arbitrarily long arithmetic progressions, that is, for every positive integer n, there exist $a \in A$ and $\delta > 0$ such that $a, a+\delta, \ldots, a+(n-1)\delta$ belong to A.

(5) Let $f : \mathbb{R} \to \mathbb{R}$ be a nonnegative function that is unbounded on every set of positive Lebesgue measure. Show that f is not measurable.

Part II. Functional Analysis

Do **three** of the following five problems.

(1) (a) Define the weak convergence of a sequence $\{x_n\}$ in a Banach space B to an element x.
(b) Show that every weak convergent sequence in a Banach space is bounded.

(2) Let $(X, ||\cdot||)$ be a Banach space. and $f, g \in X^*$ two bounded linear functionals on X. Suppose that f and g have the same null space: $N(f) = N(g)$, i.e.,

$$\{x \in X : f(x) = 0\} = \{x \in X : g(x) = 0\}.$$

Show that there is a constant λ such that $f = \lambda g$.

(3) Let $H = L^2[0,1]$ be the Hilbert space of complex valued square integrable Lebesgue measurable functions on the unit interval [0,1] and $h : [0,1] \to \mathbb{C}$ a complex valued measurable function. Define the multiplication operator $M_h f = hf$ with the domain

$$\mathscr{D}(M_h) = \{f \in H : hf \in H\}.$$

Answer the following questions and justify your answers:
(1) When is M_h a bounded operator?
(2) When is M_h a self-adjoing operator?
(3) When is M_h a compact operator?

(4) Suppose that $f \in L^2(\mathbb{R})$ whose Fourier transform $\hat{f}$ has the property that

$$\int_{\mathbb{R}} |\xi \hat{f}(\xi)|\, d\xi < \infty.$$

Show that f is continuously differentiable and its derivative f' is uniformly bounded.

(5) Suppose that F is a tempered distribution on $\mathbb{R}$ supported at the single point $x = 0$. Show that there are a positive integer N and constants c_α such that

$$F(\phi) = \sum_{|\alpha| \le N} c_\alpha \partial^\alpha \phi(0).$$

Part III. Complex Analysis

Do **three** of the following five problems.

(1) Let $f : D \to \mathbb{C}$ be holomorphic on the unit disk. Show that there is a sequence $z_n \in D$ such that $|z_n| \to 1$ and the sequence $|f(z_n)|$ is bounded.

(2) Suppose that $f_n : D \to \mathbb{C}$ is a sequence of holomorphic function on the unit disk, such that $\operatorname{Re} f_n(z) > 0$ for all n, z, and the sequence $|f_n(0)|$ is bounded. Prove or disprove that there is a subsequence f_{n_k} converging uniformly on compact sets to a holomorphic function $f : D \to \mathbb{C}$.

(3) Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic, such that $|f(z^2)| \leq |f(z)|$ for all z. Show that f is constant.

(4) (a) Find a holomorphic bijection from the upper half disk $\{z : |z| < 1, \operatorname{Im} z > 0\}$ to the upper half plane.

(b) Is the punctured disk $D \setminus \{0\}$ biholomorphic to the punctured plane $\mathbb{C} \setminus \{0\}$? Justify your answer.

(5) (a) Find a smooth function $f : D \to \mathbb{C}$ on the unit disk, extending continuously to $\bar{D}$, such that

$$\int_{\partial D} f(z)\, dz = 0,$$

but f is not holomorphic on any open subset of D.

(b) Compute the integral

$$\int_\gamma \frac{ze^z}{(z-2)(z+i)}\, dz,$$

where γ is the circle of radius 2 centered at -1, counterclockwise.