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# FALL 2018 NU PUTNAM SELECTION TEST

**Problem A1.** Suppose that a non-negative integer $n$ is the sum of two triangular numbers

$$n = \frac{a^2 + a}{2} + \frac{b^2 + b}{2}$$

with (non-negative) integers $a, b$. Write $4n+1$ into the sum of two squares, i.e., $4n+1 = x^2+y^2$ with integers $x, y$. Express $x$ and $y$ in terms of $a$ and $b$.

Show, conversely, that if $4n + 1$ is the sum of two squares, then $n$ is the sum of two triangle numbers.

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**Problem A2.** Prove that if $m, n$ are positive integers such that $\sqrt{3} > \frac{m}{n}$, then $\sqrt{3} \geq \frac{\sqrt{m^2+2}}{n}$.

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**Problem A3.** Let $P(x) = a_0 + a_1 x + \cdots + a_{n-1}x^{n-1} + x^n$, with $a_i \in \mathbb{R}$, $i \in \{0, 1, \ldots, n-1\}$, be a polynomial with roots $x_1, x_2, \ldots, x_n \in \mathbb{R}$, and let $x_{i_0} = \max_{1 \le i \le n} x_i$. Prove that if $x \ge x_{i_0}$, then $P'(x) \ge n \sqrt[n]{(P(x))^{n-1}}$.

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**Problem A4.** Define two sequences recursively as follows: $a_1 = b_1 = 1$, and for $n \geq 1$, $a_{n+1} = 2^{a_n}$, $b_{n+1} = 3^{b_n}$. Note that $a_3 = 4 > 1 = b_1^2$, $a_4 = 16 > 9 = b_2^2$, $a_5 = 65536 > 729 = b_3^2$. Prove that in general $a_{n+2} > b_n^2$ for every $n \geq 1$.

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**Problem A5.** Let $P_n$ be a regular $n$-gon ($n \geq 3$) inscribed in a unit circle (radius 1), and let $v_1, \ldots, v_n$ be its vertices. Let $d_{jk}$ be the distance between vertices $v_j$ and $v_k$. Find the sum of the squares of distances between all pairs of vertices $S = \displaystyle\sum_{1 \leq j < k \leq n} d_{jk}^2$.

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**Problem A6.** Suppose that $a_n > 0$ and $\sum a_n$ diverges. Let $s_n = \sum_{i=1}^{n} a_i$ be the partial sums. For which positive values of $p$ does the series

$$\sum_{n=1}^{\infty} \frac{a_n}{s_n^p}$$

converge?

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**Problem A7.** Let $f : [a, b] \to \mathbb{R}$ be a continuous function defined on a finite interval $[a, b]$ such that it is twice continuous differentiable and $f(a) = f(b) = 0$.

Show that there is a constant $C$ independent of $a, b$, and $f$ such that

$$\int_a^b |f(x)| \, dx \leq C \, \|f''\|_\infty (b-a)^3 \, .$$

Here $\|f''\|_\infty = \sup_{x \in [a,b]} |f''(x)|$.

(Note: Next problem asks you to prove this same inequality for a specific value of $C$).

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**Problem A8.** Let $f : [a, b] \to \mathbb{R}$ be a continuous function defined on a finite interval $[a, b]$ such that it is twice continuous differentiable and $f(a) = f(b) = 0$.

(a) Show that for every $x \in (a, b)$:

$$|f(x)| \leq \|f''\|_\infty \cdot \frac{(b-x)(x-a)}{2}.$$

Here $\|f''\|_\infty = \sup_{x\in[a,b]} |f''(x)|$.

(b) Show that

$$\int_a^b |f(x)|\, dx \leq \frac{1}{12}\|f''\|_\infty (b-a)^3.$$

(Note: This is the same as the previous problem, but now we require $C = 1/12$.)