NORTHWESTERN UNIVERSITY Tuesday, Oct 6th, 2015

NAME: ANSWERS

FALL 2015 NU PUTNAM SELECTION TEST

Problem A1. Show that $\log(1+x) > x/(1+x)$ for all $x > 0$.

- *Answer:* We have that $(\log x)' = 1/(1+x)$, and $(x/(1+x))' = 1/(1+x)^2$. Since $1/(1+x) > 1/(1+x)^2$ for $x > 0$, the function $\log x$ grows faster than $1/(1+x)$ in $(0, \infty)$. Since both functions take the same value 1 at $x = 0$, we get $\log x > x/(1+x)$ for $x > 0$, QED.

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Problem A2. Define the sequence $a_0 = 0$, $a_{n+1} = \sqrt{\frac{1+a_n}{2}}$ for $n \geq 0$. Find

$$S = \sum_{n=0}^{\infty} \arccos a_n \,.$$

(Note: $y = \arccos x \Leftrightarrow y \in [0, \pi]$ and $\cos y = x$.)

- *Answer:* The answer is π.

Proof: Define $\alpha_n = \arccos a_n$. Then $\alpha_0 = \arccos 0 = \frac{\pi}{2}$, and $\cos \alpha_{n+1} = \sqrt{\frac{1+\cos \alpha_n}{2}} = \cos \frac{\alpha_n}{2}$, hence $\alpha_{n+1} = \frac{\alpha_n}{2}$, and the sum is

$$S = \frac{\pi}{2} + \frac{\pi}{4} + \frac{\pi}{8} + \frac{\pi}{16} + \cdots = \pi \,.$$

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Problem A3. Let r be a real number in the interval $[0,1)$. Find the sum

$$S = \sum_{k=1}^{\infty} \frac{(-1)^{\lfloor 2^k r \rfloor}}{2^k},$$

where $\lfloor x \rfloor$ = integer part of x = greatest integer less that or equal to x.

- *Answer:* The answer is $S = 1 - 2r$.

Proof: Consider the binary representation of r: $r = \sum_{i=1}^{\infty} \frac{a_k}{2^k}$, where each a_i is 0 or 1. Then $\lfloor 2^k x \rfloor = a_k$, and $(-1)^{\lfloor 2^k x \rfloor} = (-1)^{a_k} = 1 - 2a_k$, hence:

$$S = \sum_{k=1}^{\infty} \frac{1-2a_k}{2^k} = \sum_{k=1}^{\infty} \frac{1}{2^k} - 2\sum_{k=1}^{\infty} \frac{a_k}{2^k} = 1 - 2r\,.$$

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Problem A4. One hundred passengers board a plane with exactly 100 seats. The first passenger takes a seat at random. The second passenger takes his own seat if it is available, otherwise he takes at random a seat among the available ones. The third passenger takes his own seat if it is available, otherwise he takes at random a seat among the available ones. This process continues until all the 100 passengers have boarded the plane. What is the probability that the last passenger takes his own seat?

- *Answer:* The answer is 1/2.

Assume passenger n is assigned seat number n, so the first passenger is assigned seat number 1, and the last passenger is assigned seat number 100. As long as seats number 1 and 100 remain unoccupied, each time a passenger picks a seat at random (either because he/she is the first passenger, or because he/she found his/her seat occupied), three possible things may happen:

(1) The passenger picks seat number 1, and from there on everybody else seats in their assigned seat, including the last passenger.

(2) The passenger picks seat number 100, and the last passenger will find his seat occupied.

(3) The passenger picks some other seat besides 1 and 100, and the question of whether the last passenger will find his seat available remains open.

As soon as (1) or (2) happens the question of whether the last passenger will take his own seat is settled, in the positive if (1) happens, and in the negative if (2) happens. Since both possibilities have the same chances, the probability of each is 1/2.

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Problem A5. Prove that the following divisibility criteria by 61 actually works. Divisibility by 61: Let n be a positive integer. Let d be the rightmost digit of n (in decimal notation), and let n' be the number obtained by removing from n its rightmost digit (if n has only one digit then $n' = 0$). Replace n with $n' - 6d$. Repeat those steps while the result is still a positive integer. If the process ends in zero then the original number is divisible by 61, otherwise it is not. Example for $n = 21045$: $2104-6\cdot5 = 2074$, $207-6\cdot4 = 183$, $18-6\cdot3 = 0$. Hence 21045 is divisible by 61.

- *Answer:* We have that $6 \cdot 10 = 60 \equiv -1 \pmod{61}$, hence if we write $n = 10n' + d$ with $0 \leq d < 10$ we have $6n = 60n + 6d \equiv 6d - n \pmod{61}$. Since 61 is a prime number not dividing 6, n is divisible by 61 if and only if $6n$ is divisible by 61, or equivalently, $6n \equiv 0 \pmod{61}$, which is also equivalent to $n' - 6d \equiv 0 \pmod{61}$. Hence n is divisible by 61 if and only if $n' - 6d$ is divisible by 61. On the other hand $n' - 6d < n$, hence the process will yield a decreasing sequence of positive numbers and can be continued only for a finite number of steps, until reaching either zero or a negative number. Now we distinguish two cases.

- *Case 1:* If the final step yields zero then all the elements of the sequence are congruent with zero modulo 61, and they are divisible by 61, including the original number.

- *Case 2:* Otherwise we will have a final step in which $n = 10n' + d > 0$, $n' \geq 0$, $0 \leq d < 10$, and $n' - 6d < 0$. Since $0 \leq d \leq 9$ we have $0 \leq 6d \leq 54$, hence $-54 \leq n' - 6d < 0$, which implies $n' - 6d \not\equiv \pmod{61}$, and the elements of the sequence are *not* divisible by 61.

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Problem A6. Flip a fair coin until heads turns out twice consecutively. What is the expected number of flips?

- *Answer:* Let e be the expected number of flips until getting two heads in a row. Start flipping the coin. If we get a tail with the first flip (probability 1/2) then the expected number of flips will be now $e+1$. If we get head, then tail (probability 1/4) then the expected number of flips will become $e+2$. If the first two flips yield heads (probability 1/4) then the number of flips will be 2. Hence:

$$e = \frac{1}{2}(e+1) + \frac{1}{4}(e+2) + \frac{1}{4} \cdot 2 = \frac{3}{4}e + \frac{3}{2}\,,$$

and solving for e we get $e = 6$.