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Problem A1. Let $a_1, a_2, \ldots, a_n$ be n not necessarily distinct integers. Prove that there exist a subset of these numbers whose sum is divisible by n.

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Problem A2. If a, b, and c are the sides of a triangle, prove that

$$\frac{a}{b+c-a} + \frac{b}{c+a-b} + \frac{c}{a+b-c} \geq 3\,.$$

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Problem A3. Does there exist a positive sequence a_n such that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} 1/(n^2 a_n)$ are convergent?

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Problem A4. On a table there is a row of fifty coins, of various denominations (the denominations could be of any values). Alice picks a coin from one of the ends and puts it in her pocket, then Bob chooses a coin from one of the ends and puts it in his pocket, and the alternation continues until Bob pockets the last coin. Prove that Alice can play so that she guarantees at least as much money as Bob.

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Problem A5. Prove that there is no polynomial $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$ with integer coefficients and of degree at least 1 with the property that $P(0)$, $P(1)$, $P(2)$,..., are all prime numbers.

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Problem A6. Given thirteen real numbers $r_1, r_2, \ldots, r_{13}$, prove that there are two of them r_p, r_q, $p \neq q$, such that $|r_p - r_q| \leq (2-\sqrt{3})|1 + r_p r_q|$. (Note: $2 - \sqrt{3} = \tan\frac{\pi}{12}$.)

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Problem A7. (Note: this question was not included in the final version of the test.) The digital root of a number is the (single digit) value obtained by repeatedly adding the (base 10) digits of the number, then the digits of the sum, and so on until obtaining a single digit—e.g. the digital root of 65,536 is 7, because $6+5+5+3+6=25$ and $2+5=7$. Consider the sequence $a_n =$ integer part of $10^n\pi$, i.e.,

$$a_1 = 31\,, \quad a_2 = 314\,, \quad a_3 = 3141\,, \quad a_4 = 31415\,, \quad a_5 = 314159\,, \quad \dots$$

and let b_n be the sequence

$$b_1 = a_1\,, \quad b_2 = a_1^{a_2}\,, \quad b_3 = a_1^{a_2^{a_3}}\,, \quad b_4 = a_1^{a_2^{a_3^{a_4}}}\,, \quad \dots$$

Find the digital root of b_{10^6}.