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WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

Problem A1. Find the integer values of x for which the following function takes integer values:

$$f(x) = \frac{x^2}{x+3}\,.$$

Answer:

The function can be rewritten like this:

$$f(x) = x - 3 + \frac{9}{x+3}\,.$$

The solutions are the integer values of x for which $9/(x+3)$ is an integer, i.e.: $x+3 =$ divisor of $9 = \pm 1, \pm 3, \pm 9$, so

$$x = -12, -6, -4, -2, 0, 6\,.$$

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Problem A2. On a table there are 100 tokens. Taking turns two players remove 5, 6, 7, 8, 9 or 10 tokens, at their choice. The player that removes the last token wins. Find a winning strategy and determine which player will be the winner.

Answer:

The first player takes 10 tokens leaving 90 tokens on the table. Then if the second player takes n tokens $5 \leq n \leq 10$, the first player takes $15 - n$, so they will remove 15 tokens together. Since 90 is a multiple of 15, the first player will take the last token and win.

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Problem A3. In a group of n people $(n \geq 2)$ each person picks another person at random and, at the sound of "now!", throws a pie to him/her. Assume that all pies have the same probability p of hitting their target, and if the pie misses its intended target it does not hit anybody else. What is the expected number of people not hit by a pie?

Answer:

A person not hit by a pie is called a "survivor". The probability that a given person A is hit by a pie thrown by some other given person B is the probability $1/(n-1)$ of being picked by that person times the probability p that the pie hits its target, so $p/(n-1)$. The probability that A is not hit by B is $1 - p/(n-1)$. The probability of A not being hit by anyone (i.e., of being a survivor) is $(1 - \frac{p}{n-1})^{n-1}$. The same is true for each of the n people in the group, so the expected number of survivors is $n\,(1 - \frac{p}{n-1})^{n-1}$.

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Problem A4. If $x \neq 0$ prove that $\dfrac{\sin x}{x} = \displaystyle\prod_{n=1}^{\infty} \cos\left(\frac{x}{2^n}\right)$.

Answer:

$$\begin{aligned}
\frac{\sin x}{x} &= \frac{\sin 2x/2}{x} \\
&= \frac{1}{x} 2 \sin\frac{x}{2}\cos\frac{x}{2} \\
&= \frac{1}{x} 2^2 \sin\frac{x}{2^2}\cos\frac{x}{2^2}\cos\frac{x}{2} \\
&\dots \\
&= \frac{2^k}{x}\sin\frac{x}{2^k}\prod_{n=1}^{k}\cos\left(\frac{x}{2^n}\right)
\end{aligned}$$

On the other hand we have: $\displaystyle\lim_{k\to\infty}\frac{2^k}{x}\sin\frac{x}{2^k} = \lim_{k\to\infty}\frac{\sin\frac{x}{2^k}}{x/2^k} = \lim_{h\to 0}\frac{\sin h}{h} = 1$,

and the limit of the product of cosines is the RHS of the desired equation.

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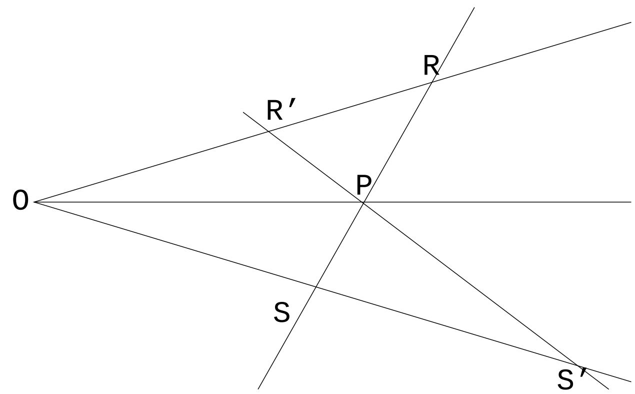
Problem A5. In the figure OP is the bisector of angle ROS. Prove than $1/|OR|+1/|OS| = 1/|OR'|+1/|OS'|$.



Answer:

Let α be the angle $\alpha = \widehat{ROP} = \widehat{POS}$. The area of triangle ROS is $\frac{1}{2}|OR||OS|\sin 2\alpha$. That area is also the sum of the areas of the triangles ROP and POS, hence:

$$\frac{1}{2}|OR||OP|\sin\alpha + \frac{1}{2}|OP||OS|\sin\alpha = \frac{1}{2}|OR||OS|\sin 2\alpha\,.$$

Dividing by $\frac{1}{2}|OR||OS||OP|\sin\alpha$ we get:

$$\frac{1}{|OR|} + \frac{1}{|OS|} = \frac{\sin 2\alpha}{|OP|\sin\alpha}\,.$$

Note that the RHS depends on α and P only, so we will get the same if we replace R and S respectively with R' and S'; consequently:

$$\frac{1}{|OR|} + \frac{1}{|OS|} = \frac{1}{|OR'|} + \frac{1}{|OS'|}\,.$$

Q.E.D.

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Problem A6. We have a calculator with two registers R_1 and R_2, and four operations:

(1) $R_1 + R_2 \to R_2$ (add the content of register R_1 to register R_2.)
(2) $-R_1 + R_2 \to R_2$ (subtract the content of register R_1 from register R_2.)
(3) $R_1 + R_2 \to R_1$ (add the content of register R_2 to register R_1.)
(4) $R_1 - R_2 \to R_1$ (subtract the content of register R_2 from register R_1.)

For instance, if $R_1 = x$ (register R_1 contains the number x) and $R_2 = y$ (R_2 contains y), after applying the operation $R_1 + R_2 \to R_2$ we end up with $R_1 = x$ and $R_2 = x + y$. Assume that initially we have $R_1 = x$ and $R_2 = y$, where x and y are arbitrary numbers. For each of the following tasks describe a sequence of operations that would allow us to perform the task, or prove that it is impossible:

(1) Swap the contents of registers R_1 and R_2 changing the sign of y in the process, so we would end up with $R_1 = -y$, $R_2 = x$.
(2) Swap the contents of registers R_1 and R_2, so that we would end up with $R_1 = y$, $R_2 = x$.

Answer:

(1) The answer to the first question may be the following sequence of operations:

$$\begin{aligned} &R_1 + R_2 \to R_2 \quad (R_1 = x, R_2 = x + y) \\ &R_1 - R_2 \to R_1 \quad (R_1 = -y, R_2 = x + y) \\ &R_1 + R_2 \to R_2 \quad (R_1 = -y, R_2 = x) \end{aligned}$$

(2) Regarding the second question, we will see that the task is impossible. If we represent the initial content of the registers with a vector $\begin{pmatrix} x \\ y \end{pmatrix}$, then each of the given operations consists of multiplying to the left by each of the following matrices respectively:

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

Note that all those matrices have determinant equal $+1$. On the other hand the desired task is equivalent to multiplying by the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} .$$

The determinant of this matrix is -1, so it is impossible to obtain it by multiplying the given matrices (recall that the determinant of a product of square matrices is the product of the determinants of the factors). Hence the task is impossible.

An equivalent approach consists of noting that the result of applying any number of operations to $R_1 = x$, $R_2 = y$ always yields a result of the form $R_1 = ax + by$, $R_2 = cx + dy$ with $ad - bc = 1$, and for $R_1 = y$, $R_2 = x$ we would have $ad - bc = -1$.