Topology Preliminary Exam, February 1997

Do 6 of 8 problems. Be sure to indicate which problems you are submitting.

(1) Let M be a smooth (C^∞) compact n dimensional submanifold of R^{2n+2}.

(a) Prove that there exists a $2n+1$ dimensional linear subspace H of R^{2n+2} such that the orthogonal projection $P : R^{2n+2} \to H$ has the property that P restricted to M is one-to-one.

(b) Prove that, in addition to the property from part (a), H can be chosen so that for every $x \in M$, the derivative $dP_x : TM_x \to TH_{P(x)}$ is one-to-one, that is, so P defines an embedding of M into H.

(2) Show that there does **not** exist a continuous function $f : \mathbf{C}P^2 \to S^2 \times S^2$ which induces an isomorphism of integral homology groups in dimension four:

$$f_* : H_4(\mathbf{C}P^2; \mathbf{Z}) \xrightarrow{\approx} H_4(S^2 \times S^2; \mathbf{Z}).$$

(3) Let K be a knot in $\mathbf{R}^3$, that is, a subset of $\mathbf{R}^3$ which is homeomorphic to the circle (or 1-sphere) S^1. View the 3-sphere S^3 as the one-point compactification of R^3.

(a) Show that $R^3 \setminus K$ is path connected.

(b) Show that the inclusion $i : R^3 \setminus K \to S^3 \setminus K$ induces an isomorphism of fundamental groups.

(4) Let D^n be the unit disk in $\mathbf{R}^n$ and S^{n-1} its boundary, the unit $(n-1)$–sphere. Suppose $f : D^n \to \mathbf{R}^n$ is a continous function which the identity on S^{n-1} and is locally one–to–one on the *interior* of D^n.

Show that f maps the interior of D^n onto the interior of D^n.

Comment: f is also one–to–one, but you need not prove this.

(5) Give an example of two connected finite CW–complexes K and L which are not acyclic and such that the inclusion of the wedge $K \vee L$ into the Cartesian product $K \times L$ induces and isomorphism of integral homology groups

$$f_* : H_n(K \vee L; \mathbf{Z}) \xrightarrow{\approx} H_n(K \times L; \mathbf{Z})$$

in all dimensions $n \geq 0$. Explain *briefly* why your example satisfies the requirement.

(6) Let K be a finite CW–complex of dimension n. Let α_p be the number of p–cells of K and let β_p be the p–th betti number, the rank of $H_p(K; \mathbf{Z})$. Give a proof that

$$\sum_{p=0}^{n} (-1)^p \alpha_p = \sum_{p=0}^{n} (-1)^p \beta_p.$$

Remark: Of course, this number is the Euler Characteristic $\chi(K)$.

(7) Let M be a compact connected n–manifold such that $H_1(M; \mathbf{Z}/2\mathbf{Z}) = 0$. Prove that M is orientable.

(8) Let $p : S^n \to \mathbf{R}P^n$ be the standard covering map of the real projective plane of dimension n by the n–sphere. Prove that p is not homotopic to a constant map.