Geometry and topology preliminary exam, September 2011.

Question 1.
Let $Gr(2,4)$ denote the Grassmannian of two-dimensional planes in $\mathbb{R}^4$. Let $GL_4(\mathbb{R})$ be the general linear group of invertible linear transformations of $\mathbb{R}^4$. Let $P = \langle e_1, e_2 \rangle \in Gr(2,4)$ be the two-dimensional plane spanned by the basis vectors $e_1, e_2 \in \mathbb{R}^4$.

(1) Find an open neighborhood $U \subset Gr(2,4)$ containing P, and which is homeomorphic to an open subset of some $\mathbb{R}^n$.
(2) Show $GL_4(\mathbb{R})$ acts transitively on $Gr(2,4)$. What is the stabilizer of P?
(3) Show $Gr(2,4)$ is a smooth manifold.

Let $X \subset Gr(2,4)$ denote the subspace of planes $Q \in Gr(2,4)$ such that $\dim(Q \cap P) \geq 1$.

(4) Show that the complement $Gr(2,4) \setminus X$ is contractible.
(5) Find equations for the intersection of X with your open neighborhood $U \subset Gr(2,4)$ from part 1.
(6) Show X is not a smooth manifold at P.

Question 2.
Consider the two-dimensional sphere $S^2 = \mathbb{C} \cup \{\infty\}$.

Let $0 \in \mathbb{C} \subset S^2$ denote zero.

Let $\mathbb{Z}/n\mathbb{Z} \subset \mathbb{C}^\times$ be the cyclic subgroup of nth roots of unity.

Let $X_n = S^2 \setminus \mathbb{Z}/n\mathbb{Z}$ denote the complement of the nth roots of unity .

(1) For $n = 1, 2, 3, \ldots$, calculate $\pi_1(X_n, 0)$ in terms of generators and relations.
(2) For $n = 2$, why is there a canonical isomorphism $\pi_1(X_2, 0) \simeq \pi_1(X_2, x)$ for any $x \in X_2$?
(3) For $n = 2$, observe that $\mathbb{Z}/3\mathbb{Z} \subset \mathbb{C}^\times$ acts on X_3 by multiplication. Calculate the induced action on $\pi_1(X_3, 0)$ in terms of generators and relations.
(4) For $n = 4$, classify all (not necessarily connected) two-fold covers of X_4.

Question 3.
Let

$$X = \{(x,y) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid \langle x, y \rangle = 0,\ \|x\| = 1,\ \ \|y\| = 1\} \subset S^2 \times S^2$$

(where $\langle x, y \rangle$ denotes the usual Euclidean inner product on $\mathbb{R}^3$).

Use the Mayer-Vietoris sequence to compute the cohmology of X.

You may use any results you know about the cohomology of spheres and tori, as long as they are stated precisely.

Question 4.
Let M be a smooth compact oriented manifold, and let $f : M \to M$ be a diffeomorphism. A point $p \in M$ is a fixed point of f if $f(p) = p$. We say that f is regular if the derivative $D_p f$ of f at p, which is a linear automorphism of T_pM, has no fixed points.

For each fixed point p, define a number $L(p)$ to be 1 if $\det D_p f > 0$, and -1 if $\det D_p f < 0$.

Let $\Gamma_f \subset M \times M$ be the graph of f, which is the image of the embedding

$$\begin{aligned} \mathrm{Id} \times f : M &\to M \times M \\ x &\mapsto (x, f(x)). \end{aligned}$$

Note that Γ_f is naturally oriented, because M is.

Let $\triangle \subset M \times M$ be the diagonal, that is, the graph of the identity map.

(1) Show that the set of fixed points of f is the intersection of Γ_f with $\triangle$.
(2) Show that f is regular if and only if Γ_f intersects $\triangle$ transversely.
(3) Let $[\Gamma_f]$ and $[\triangle]$ denote the fundamental classes of these submanifolds of $M \times M$. Supposing that f is regular, use intersection theory to prove that

$$\sum_{p \in \mathrm{Fix}(f)} L(p) = \int_{M \times M} [\Gamma_f] \wedge [\triangle].$$

(4) Deduce that the Lefschetz number $L(f) = \sum_{p \in \mathrm{Fix}(f)} L(p)$ of a regular diffeomorphism only depends on the smooth homotopy class of f.
(5) Prove that every diffeomorphism f of S^2 which is homotopic to the identity has at least one fixed point.

Question 5.
Define a submanifold $H \subset \mathbb{R}^9$ as follows: $H_3 \subset GL(3, \mathbb{R}) \subset Mat(3 \times 3, \mathbb{R}) \cong \mathbb{R}^9$ is given by the upper-triangular 3×3 matrices with 1's along the diagonal. Note $H \cong \mathbb{R}^3$, coordinatized by writing $h \in H$ as

$$h = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}.$$

Note, too, that H is a closed subgroup of a Lie group, therefore a Lie group itself.

A basis for the tangent space T_eH to H at the identity at the identity $e = (x = 0, y = 0, z = 0)$ is given by the three vectors $\partial_x|_e$, $\partial_y|_e$, $\partial_z|_e$.

Let U_e and V_e denote elements of T_eH; we can write U_e and V_e as

$$\begin{aligned} U_e &= a\partial_x|_e + b\partial_y|_e + c\partial_z|_e \\ V_e &= a'\partial_x|_e + b'\partial_y|_e + c'\partial_z|_e \end{aligned}$$

for some $a, a', b, b', c, c' \in \mathbb{R}$.

- Use the definition of left-invariance and the Lie group structure to write down the left-invariant vector *fields* U and V corresponding to U_e and V_e.

In this way, you have constructed a map $\Psi : T_eH \to Lie(H)$, where $Lie(H)$ is the set of left-invariant vector fields.

- We can think of the vectors $U_e, V_e \in T_eH$ as strictly upper triangular 3×3 matrices, in an evident way. Thus, we can define the matrix product $U_e \cdot V_e$ and $V_e \cdot U_e$. Verify that
$$\Psi(U_e \cdot V_e - V_e \cdot U_e) = [U, V],$$
i.e. that Ψ intertwines the commutator bracket with the Lie bracket.

Question 6.
Let M be a manifold and let $\pi : T^*M \to M$ be the cotangent bundle with fibration map π. Define the *canonical one-form* on T^*M (a one-form on a bundle which is itself a space of one-forms) by the formula
$$\Theta_\xi(v) = \xi(\pi_* v).$$
Here ξ is a point of T^*M and $v \in T_\xi(T^*M)$ is a tangent vector at ξ. Now let $M = \mathbb{R}^2$ and let (x, y, ξ_1, ξ_2) coordinatize the cotangent bundle $T^*\mathbb{R}^2 \cong \mathbb{R}^4$ consisting of covectors $\xi_{(x,y)} = \xi_1 dx|_{(x,y)} + \xi_2 dy|_{(x,y)}$.

- Verify that in these coordinates, the one-form Θ defined above is $\Theta = \xi_1 dx + \xi_2 dy$.

Now switch to polar coordinates (r, θ) in the fibers, $\xi_1 = r\cos\theta$, $\xi_2 = r\sin\theta$, and define the unit tangent bundle $S^* = \{r = 1\}$, with $i : S^* \hookrightarrow T^*\mathbb{R}^2$ the inclusion map.

- Using the coordinates (x, y, θ), write down $i^*\Theta$, i.e. the restriction of Θ to S^*.

Let us define the one-form $\alpha = i^*\Theta$ on S^* to be the form you wrote down above. With these definitions, a curve $C \hookrightarrow S^*$ in S^* is said to be *Legendrian* if the restriction of α to the curve is zero. Now let us use the Euclidean metric on $\mathbb{R}^2$ so that we may identfy the unit tangent bundle $S \subset T\mathbb{R}^2$ and the unit cotangent bundle S^*, and we will continue to use the coordinates (x, y, θ). A parametrized curve $(x(t), y(t))$ in $\mathbb{R}^2$ induces a curve in $S \cong S^*$ by setting $\theta(t) = \tan^{-1}(y'(t)/x'(t) - \pi/2) = -\cot^{-1}(y'/x')$ to be the normal direction (effected here by the shift by $\pi/2$).

- Verify that the parametrized cusp
$$x(t) = t^2, \qquad y(t) = t^3, \qquad \theta(t) = -\cot^{-1}(y'/x') \qquad t \in \mathbb{R}$$
in $\mathbb{R}^2$ induces a Legendrian curve in S for *all* t.