PRELIMINARY EXAM IN GEOMETRY AND TOPOLOGY SPRING 2020

Instructions:

(1) There are **three** parts to this exam: I (Differentiable Topology), II (Algebraic Topology), and III (Differentiable Geometry). There are **five** problems in each part. You should present solutions to **three** problems from each part: if you present solutions to more than three problems in a part, the grader will select which three solutions contribute most to the total grade.

(2) In each problem, full credit requires proving that your answer is correct. You may quote and use theorems and formulas. If a problem asks you to state or prove a theorem or a formula, you need to provide the full details. If it asks you to disprove a statement, a counterexample will suffice, again of course with full details.

Part I. Differentiable Topology

(1) Consider the Grassmannian of complex k-planes in $\mathbb{C}^n$. (Recall that, as a space, $\mathsf{Gr}_k(\mathbb{C}^n)$ is topologized as the quotient of the Stiefel manifold $\mathsf{V}_k(\mathbb{C}^n)$ of orthonormal k-frames in $\mathbb{C}^n$, where the map $\mathsf{V}_k(\mathbb{C}^n) \to \mathsf{Gr}_k(\mathbb{C}^n)$ sends a k-frame to the k-dimensional subspace it spans.)

  (a) Is $\mathsf{Gr}_k(\mathbb{C}^n)$ compact or noncompact? Prove your answer.

  (b) What is the dimension of $\mathsf{Gr}_k(\mathbb{C}^n)$? Prove your answer.

(2) The oriented Grassmannian $\widetilde{\mathsf{Gr}}_k(\mathbb{R}^n)$ of k-planes in $\mathbb{R}^n$ is the set of oriented k-dimensional linear subspacesof $\mathbb{R}^n$, topologized as the quotient of the Stiefel manifold of k-frames $\mathsf{V}_k(\mathbb{R}^n)$. Calculate the Euler characteristic of the oriented Grassmannian $\widetilde{\mathsf{Gr}}_{n-1}(\mathbb{R}^n)$ for all n.

(3) (a) Let M be an odd-dimensional compact manifold with boundary ∂M. Prove that the Euler characteristic of the boundary is double that of M:

$$2 \cdot \chi(M) = \chi(\partial M)$$

  (b) Assuming the above, give an example of a $2n$-dimensional manifold N which is not the boundary of any $(2n+1)$-dimensional manifold M.

(4) Prove that a function $M \to \mathbb{R}$ is Morse if and only if $df : M \to T^*M$ is transverse to the zero-section.

(5) Let $f(x_0, x_1, \dots, x_n)$ be a degree d homogenous polynomial in $n+1$ complex variables over the complex numbers $\mathbb{C}$. Prove that the image of the zero-set of f defines a smooth submanifold of $\mathbb{C}\mathrm{P}^n$.

Part II. Algebraic Topology

(1) Let $T^n = S^1 \times \cdots \times S^1$ be the product of the circle with itself n-times. What is the fundamental group of T^n? What is the universal cover of T^n? Suppose X is a CW complex with finite fundamental group. Show any continuous map $X \to T^n$ is null-homotopic.

(2) The torus $T = T^2$, embedded in $\mathbb{R}^3$ in the standard way, bounds a compact region R . Two copies of R, glued together by the identity map between their boundary surfaces T, form a closed 3-manifold X . Compute the cohomology groups $H^*(X, \mathbb{Z})$ via the Mayer-Vietoris sequence for this decomposition of X into two copies of R. Now use Poincaré duality to compute the cohomology ring.

(3) Let n be an even number and $S^n \vee S^n$ the one-point union of two n-spheres. Let $\nabla : S^n \vee S^n \to S^n$ be the unique continuous map which is the identity of each copy on S^n and let $f : S^{2n-1} \to S^n \vee S^n$ be the attaching map needed for the standard CW decomposition of $S^n \times S^n$. Now let $h = \nabla \circ f : S^{2n-1} \to S^n$ and define

$$X = S^n \cup_h D^{2n}$$

to be the space obtained by attaching a $2n$-cell using h. Calculate the cohomology ring $H^*(X, \mathbb{Z})$. Note there is a continuous map $S^n \times S^n \to X$.

(4) Let $\mathbb{C}\mathrm{P}^n$ be complex projective space. Show there is an orientation reversing homeomorphism $f : \mathbb{C}\mathrm{P}^n \to \mathbb{C}\mathrm{P}^n$ if and only if n is odd.

(5) Let X be a topological space and $C_\bullet(X)$ the singular chain complex of X. Let $\varphi : C_\bullet(X) \to C_\bullet(X)$ be any **natural** chain map. Show that there is an integer n so that φ is chain homotopic to multiplication by n.

Part III. Differential Geometry

(1) Let M be a compact, connected orientable manifold of dimension $n \geq 2$, and $p \in M$. Suppose you know the de Rham cohomology groups of M, determine those of $M \setminus \{p\}$.

(2) (a) Let M be a smooth connected manifold of dimension $2n$. We say that a 2-form ω on M is symplectic if $d\omega = 0$ and $\omega \wedge \ldots \wedge \omega$ is a nowhere vanishing $2n$-form on M. Show that if M is compact with no boundary then no symplectic form ω on M is exact.

(b) Conclude that spheres S^{2n} of dimension $2n$, $n > 1$, do not admit symplectic forms.

(3) Let (M, g) be a Riemannian manifold and ∇ be the Levi-Civita connection associated to g.

(a) Define the covariant derivative D associated to ∇.

(b) Show that if V, W are vector fields along a smooth curve γ then

$$\frac{d}{dt}\langle V, W\rangle = \langle \frac{DV}{dt}, W\rangle + \langle V, \frac{DW}{dt}\rangle.$$

(c) Let X, Y be vector fields, $p \in M$ and $\gamma : [a, b] \to M$ a curve such that $\gamma'(t_0) = X(p)$, $t_0 \in (a, b)$. Show that

$$\nabla_X Y(p) = \frac{d}{dt} P^{-1}_{\gamma,t_0,t}(Y(\gamma(t)))\Big|_{t=t_0},$$

where $P_{\gamma,s,t} : T_{\gamma(s)}M \to T_{\gamma(t)}M$ is the parallel transport along γ from s to t.

(4) (a) State the Hopf-Rinow theorem concerning the relation between completeness and the exponential maps of a Riemannian manifold.

(b) Suppose M is a complete Riemannian manifold. Show that M is compact if and only if the diameter of M

$$\text{diam}(M) = \sup\{d(p, q) : p, q \in M\}$$

is finite.

(5) Let $\gamma : [a, b] \to M$ be a curve on a Riemannian manifold M.

(a) Write down the definition of the energy $E(\gamma)$.

(b) Let V be a smooth vector field along γ. Consider a smooth variation of γ given by $F : [a, b] \times (-\epsilon, \epsilon) \to M$ with $F(t, 0) = \gamma(t)$ and $\frac{\partial}{\partial s}\gamma(t, s) = V$. Write $\gamma_s(t) = F(t, s)$. Assume that $F(a, s) = \gamma(a)$ and $F(b, s) = \gamma(b)$ for all $s \in (-\epsilon, \epsilon)$. Show that

$$\frac{\partial}{\partial s}E(\gamma_s)\Big|_{s=0} = \int_a^b \langle V, \frac{D}{dt}\frac{d\gamma}{dt}\rangle dt.$$

(c) Assume that γ is a critical point of E. Conclude that $\frac{d\gamma}{dt}$ is parallel along γ.