PRELIMINARY EXAM IN TOPOLOGY & GEOMETRY FALL 2022

INSTRUCTIONS:

(1) There are **three** parts to this exam: I Analysis on Manifolds, II Introduction to Algebraic Topology, and III Cohomology. Do **three** problems from each part. If you attempt more than three problems in each section, then the three problems with highest scores will count towards your overall performance.

(2) In each problem, full credit requires proving that your answer is correct. You may quote and use theorems and formulas. But if a problem asks you to state or prove a theorem or a formula, you need to provide the full details.

Part I. Analysis on Manifolds

Do **three** of the following five problems.

Problem 1. (a) State the following theorems: (1) the inverse function theorem; (2) the implicit function theorem; (3) Sard's theorem. (b) Let $F : \mathbb{R}^2 \to \mathbb{R}$ be a smooth map near the origin (0, 0), $F(0,0) = 0$ and $DF(0,0) \neq 0$. Show that it is possible to choose local charts on $\mathbb{R}^2$ and $\mathbb{R}$ such that in the new charts the smooth map is simply the projection $F(u,v) = u$ near the origin.

Problem 2. Let $(\mathbb{R}^2, \nabla)$ be the euclidean plane with its usual Levi-Civita connection. Let (r, θ) be the usual polar coordinates on the euclidean plane (excluding the origin). (a) Calculate the Christoffel symbols of the Levi-Civita connection ∇ on $\mathbb{R}^2$ of the local coordinates (r, θ); (b) Find the general equation of geodesics on $\mathbb{R}^2$ in terms of polar coordinates (r, θ).

Problem 3. (a) Define the Lie bracket of two vector fields on a smooth manifold. (b) Let X, Y be two independent vector fields on $\mathbb{R}^2$ such that $[X, Y] = 0$ in a neighborhood of the origin (0, 0). Show that it is possible to choose local coordinates (x, y) near the origin such that $X = \partial/\partial x$ and $Y = \partial/\partial y$.

Problem 4. (a) Give a definition of a differential form θ on a smooth manifold M. (b) Give a definition of the exterior differentiation $d\theta$ of a differential form. (c) State the Stokes' theorem on an oriented manifold M with boundary. Now let M be a smooth manifold equipped with a connection $\nabla : \Gamma(M) \times \Gamma(M) \to \Gamma(M)$, where $\Gamma(M)$ is the space of vector fields on M. (d) Describe how to extend the connection from vector fields to tensor fields $\nabla : \Gamma(M) \times \Gamma^{p,q}(M) \to \Gamma^{p,q}(M)$. (e) Suppose that θ is a 2-form on M. Show that

$$(\nabla_X \theta) = X\theta(Y, Z) - \theta(\nabla_X Y, Z) - \theta(Y, \nabla_X Z).$$

(f) Suppose that the connection is torsion free and that η is a 1-form on M . Show that

$$(d\theta)(X, Y) = (\nabla_X \theta)(Y) - (\nabla_Y \theta)(X).$$

Problem 5. (a) State Stokes' theorem for a oriented smooth manifold with boundary. (b) Suppose that M is an oriented Riemannian manifold with boundary and X a vector field on M. Define the divergence $\mathrm{div}X$ of the vector field M and using Stokes' theorem or otherwise to show Gauss' theorem:

$$\int_M (\mathrm{div}X)\, dm = \int_{\partial M} \langle X, N\rangle \, d\sigma,$$

where N is the outward unit normal vector of the boundary and dm and $d\sigma$ are the Riemannian volume measure on M and ∂M. Note: You can assume $\mathrm{dim}M = 2$ if it simplifies your proof.

Part II. Introduction to Algebraic Topology.

Do **three** of the following five problems.

Problem 1. Let $\mathrm{GL}_n(\mathbf{C})$ denote the group of $n \times n$ invertible complex matrices endowed with the subspace topology from the inclusion $\mathrm{GL}_n(\mathbf{C}) \subseteq \mathbf{C}^{n^2}$. Compute $\pi_1(\mathrm{GL}_n(\mathbf{C}))$.

Problem 2. Prove the fundamental theorem of algebra using the fundamental group: every polynomial $f(z) = z^n + a_1 z^{n-1} + \cdots + a_n$ has a root in the complex numbers.

Problem 3. Let G be a group. Construct a CW complex X with a basepoint x such that $\pi_1(X, x) \cong G$.

Problem 4. Let X be $S^2 \setminus \{x, y\}$ where x and y are distinct points. (a) Compute $\pi_1(X)$. (b) Compute $\mathrm{H}_*(X, \mathbb{Z})$.

Problem 5. (a) State the excision theorem for homology. (b) Using the excision theorem for homology, compute $\mathrm{H}_n(S^n, \mathbf{Z})$ for all non-negative integers n.

Part III. Cohomology.

Do **three** of the following five problems.

Problem 1. Prove the Poincaré lemma: if $p > 0$, every closed differentiable p-form ω on $\mathbf{R}^n$ is exact: $\omega = \mathrm{d}\tau$ for some τ.

Problem 2. Let N_g be the non-orientable closed surface of genus g. Compute (with justification for the product structure) the singular cohomology ring $\mathrm{H}^*(\mathrm{N}_g, \mathbb{Z})$.

Problem 3. Let X be a topological spaces. (a) Prove that the category $\mathcal{P}_{\mathrm{Ab}}(X)$ of presheaves of abelian groups on X is an abelian category. (b) Prove that the category $\mathrm{Shv}_{\mathrm{Ab}}(X)$ of sheaves of abelian groups on X is an abelian category.

Problem 4. Compute the singular cohomology ring $\mathrm{H}^*(S^2 \vee S^2 \vee S^2, \mathbf{Z})$.

Problem 5. Suppose that X is a connected topological manifold and that $\pi_1(X)$ is a simple group not isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Prove that X is orientable.