PRELIMINARY EXAM IN GEOMETRY AND TOPOLOGY FALL 2020

INSTRUCTIONS: (1) There are **three** parts to this exam. Do **three** problems from each part. If you attempt more than three, then indicate which you would like graded; otherwise we will grade the first three you attempt in each section.

(2) In each problem, full credit requires proving that your answer is correct. You may quote and use theorems and formulas. But if a problem asks you to state or prove a theorem or a formula, you need to provide the full details.

Part I. Differentiable Topology

Do **three** of the following five problems.

(1) Provide the following statements.
  (a) Give a complete definition of: a topological manifold M.
  (b) Give a complete definition of: a smooth manifold M.
  (c) Give a complete definition of: a diffeomorphism between smooth manifolds $f : M \to N$.
  (d) State the inverse function theorem.

(2) Consider the subspace $V \subset \mathbb{R}^4$ of solutions to the equation
$$x_1^4 - x_2 x_3 x_4 = 1 \,.$$
Is V a smooth manifold? Prove your answer.

(3) List those finite groups which act freely on S^2. (That is, list those G for which there exists a group action $G \times S^2 \to S^2$ such that $gx \neq hx$ for $g \neq h$.) Prove that your list is complete.

(4) Let M be a compact smooth n-manifold (without boundary) which is homotopy equivalent to a point. Prove that M is a point.

(5) Consider the vector field $v = x\partial_x$ on $M = \mathbb{R}_{>0}$. Prove that v generates a 1-parameter group of diffeomorphisms $\varphi_t : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$. In your proof, write down φ_t explicitly. Find a Riemannian metric g on $\mathbb{R}_{>0}$ for which φ_t is an isometry for all $t \in \mathbb{R}$.

Part II. Algebraic Topology

Do **three** of the following five problems.

(1) Let G be a finitely presented group. Show there is a based CW complex with fundamental group G.

Recall that G is finitely presented if it is a quotient of a finitely generated free group F by a normal subgroup $N = \langle a_1, \ldots, a_n \rangle$ with $a_i \in F$. Thus N is the smallest normal subgroup containing the a_i.

(2) Let S^n denote the standard n-sphere in $\mathbb{R}^{n+1}$ and $S^{n-1} \subseteq S^n$ by the inclusion as the first n-components. Let $S^\infty = \cup_n S^n$ with colimit topology: $C \subseteq S^\infty$ is closed if and only if $C \cap S^n$ is closed for all n. Let X be a finite CW-complex.

(a) Prove every map $X \to S^\infty$ is null-homotopic.

(b) Let $a \in H^1(\mathbb{RP}^\infty, \mathbb{Z}/2) \cong \mathbb{Z}/2$ be the generator. Prove the map $[X, \mathbb{RP}^\infty] \to H^1(X, \mathbb{Z}/2)$ sending f to $f^*(a)$ is an injection.

(c) Now prove the same map $[X, \mathbb{RP}^\infty] \to H^1(X, \mathbb{Z}/2)$ is a surjection.

(3) Show that the spaces $X = (S^1 \times \mathbb{CP}^\infty)/(S^1 \times \{*\})$ and $Y = S^3 \times \mathbb{CP}^\infty$ have isomorphic cohomology rings over $\mathbb{Z}$.

Note: X and Y are *not* homotopy equivalent.

(4) Let M be a closed (compact, without boundary) 3-manifold and suppose M cannot be oriented. Write $H_1(M, \mathbb{Z}) \cong \mathbb{Z}^n \times T$ where T is a torsion abelian group. Prove $H_2(M) \cong \mathbb{Z}^{n-1} \times \mathbb{Z}/2$.

(5) Let x and y be any two distinct point in $\mathbb{RP}^3$, and consider the complement $\mathbb{RP}^3 \setminus \{x, y\}$. Compute:

(a) the fundamental group $\pi_1(\mathbb{RP}^3 \setminus \{x, y\})$;

(b) the cohomology ring $H^*(\mathbb{RP}^3 \setminus \{x, y\}, \mathbb{Z})$.

Part III. Differential Geometry

Do **three** of the following five problems.

(1) Consider the complex projective space $\mathbb{C}\mathrm{P}^2$ of complex lines in $\mathbb{C}^3$.
   (a) Give a complete enumeration of 2-dimensional complex vector bundles on $\mathbb{C}\mathrm{P}^2$.
   (b) In your list, identify the tangent bundle $T\mathbb{C}\mathrm{P}^2$.

In the following two problems, consider the upper half plane $\mathbb{H} = \{(x_1, x_2) \in \mathbb{R}^2 | x_2 > 0\}$ with the metric

$$g_{11} = g_{22} = \frac{1}{x_2^2}$$
$$g_{12} = 0 \,.$$

(2) (a) Write the equation for the Christoffel symbols Γ_{ij}^k of the associated Levi-Civita connection as a function of the metric g_{ij}.
   (b) Calculate the Christoffel symbols Γ_{ij}^k for the associated Levi-Civita connection on $\mathbb{H}$.

(3) (a) Write the geodesic equation in terms of the Christoffel symbols, expressing the condition for a smooth parametrized curve $\gamma(t) = (u_1(t), \dots, u_n(t)$ to be a geodesic.
   (b) Prove that for each constant r, the semi-circle of radius r parametrized as

$$x_1(t) = -r \cdot \mathsf{tanh}(t)$$
$$x_2(t) = r \cdot \mathsf{sech}(t)$$

   is a geodesic in $\mathbb{H}$.[1]

(4) Consider the 3-sphere S^3.
   (a) Construct a flat connection on S^3.
   (b) Construct a connection on S^3 which is not flat.
   Prove your answers.

(5) (a) Let $\omega = -ydx + xdy$. Show that ω is non-vanishing on S^1 and

$$\int_{S^1} \omega = 2\pi.$$

   (b) Define the map $\phi : \Lambda^1(S^1) \to \mathbb{R}$

$$\phi(\omega) = \int_{S^1} \omega.$$

   Show that ϕ is linear and onto and that $ker(\phi) = \{\omega \in \Lambda^1(S^1) : \omega \text{ is exact}\}$.
   (c) Use parts (a) and (b) to compute $H^1(S^1)$.

[1]Recall $\mathsf{tanh}(t) = \frac{e^t - e^{-t}}{e^t + e^{-t}}$ and $\mathsf{sech}(t) = \frac{2}{e^t + e^{-t}}$.