Preliminary Examination in Real Analysis **September 1994**

Instructions: Your have **three** hours. Do **one** problem in each part, **five** problems in all. If you try more than one problem in any part, indicate which one is to be graded.

PART I

1. State three important properties that hold for the Lebesgue integral, but do **not** hold for the Riemann integral. Illustrate with examples.
2. Show that a countable set of real numbers has Lebesgue measure zero. Can a set of Lebesgue measure zero have more than countably many points? Justify your answer with full details and proofs.

PART II

3. Show that if $(X, \mathcal{M}, \mu)$ is a measure space and $A_n \in \mathcal{M}, n = 0, 1, 2, \ldots$, then
$$\mu\Big(\cup_{n=1}^{\infty} A_n \Big) = \lim_{m\to\infty} \mu\Big(\cup_{n=1}^{m} A_n \Big).$$
4. Prove that if $f \in L^p[0,1], 1 \leq p < \infty$, and $\epsilon > 0$, then there is a step function φ such that $\|f - \varphi\|_p < \epsilon$.

PART III

5. Prove that a normed linear space X is complete if and only if every absolutely summable series is summable.
6. Prove that a linear operator from a normed vector space X into a normed vector space Y is bounded if and only if it is continuous at one point.

PART IV

7. (a) State the Baire Category Theorem.
   (b) Is the set I of irrational numbers an F_σ set? Justify your answer.
8. (a) State the Open Mapping Theorem (for Banach spaces) and the Closed Graph Theorem.
   (b) Let X be a linear vector space that is complete in the norms $\| \ \|$ and $||| \ |||$. Prove that if there is a constant C such that $\|x\| \leq C|||x|||$ for all $x \in X$, then the norms are equivalent.

PART V

9. (a) State Tonelli's Theorem.
   (b) Prove that if $f \in L^1(0,1)$ and $a > 0$, then the integral
$$F_a(x) = \int_0^x (x-y)^{a-1} f(y)\, dy$$
   exists for almost every $x \in (0,1)$ and $F_a \in L^1(0,1)$.
10. A function f is said to satisfy a Lipschitz condition on an interval if there is a constant C such that $|f(x) - f(y)| \leq C|x-y|$ for all x and y in the interval.
    (a) Show that a function satisfying a Lipschitz condition is absolutely continuous.
    (b) Show that an absolutely continuous function f satisfies a Lipschitz condition if and only if $|f'|$ is bounded.