ANALYSIS PRELIMINARY EXAM, SEPTEMBER 2010.

Part I. Do **three** of the following five problems

(1) (a) State Fatou's lemma and the monotone convergence theorem.
(b) Show that the inequality in Fatou's lemma may be strict.
(c) Show that the monotone convergence theorem need not hold for *decreasing* sequences of functions.

(2) Let $\{f_n\}$ be a sequence of Lebesgue measurable functions defined on a measurable set $E \subset \mathbb{R}$. Define E_0 to be the set of points x in E such that $\lim_{n\to\infty} f_n(x)$ exists and is finite. Prove that E_0 is measurable.

(3) Assume that $f \in L^2([0,1])$ and define

$$g(x) = \frac{1}{x}\int_0^x f(s)\, ds.$$

Show that $g \in L^2[0,1]$ and $\|g\|_2 \le 2\|f\|_2$.

(4) Let $(X, \mathscr{A}, \mu)$ and $(Y, \mathscr{B}, \nu)$ be σ-finite measure spaces, and let $f \in L^1(X \times Y, \mathscr{A} \otimes \mathscr{B}, \mu \times \nu)$. Suppose that for μ-a.e. $x \in X$:

$$\int_Y \left| f(x,y) - \int_X f(w,y)\, d\mu(w) \right| d\nu(y) = 0,$$

and for ν-a.e. $y \in Y$:

$$\int_X \left| f(x,y) - \int_Y f(x,z)\, d\nu(z) \right| d\mu(x) = 0.$$

Prove that f is $(\mu \times \nu)$ - a.e. equal to a constant function.

(5) Let $f \in L^1(X, d\mu)$ and consider the measure $\nu = f d\mu$. Show that for any $\epsilon > 0$ there is a $\delta > 0$ with the following property: for any measurable set $E \subset X$ such that $\mu(E) \le \delta$ we have $|\nu(E)| \le \epsilon$.

Part II. Do **three** of the following four problems.

(1) Is the unit ball of $L^1(X, \mu)$ weakly compact? If yes, prove it. If not, find a counterexample.

(2) Suppose that A is a linear operator defined everywhere on a Hilbert space H satisfying $\langle Av, w\rangle = \langle v, Aw\rangle$ for all $v, w \in H$. Show that A is continuous.

(3) Let χ be the characteristic function of the rational numbers in $[0,1]$ (one on the rationals, zero on the irrationals). Does there exist a sequence $\{f_n\}$ of continuous functions on $[0,1]$ that converges pointwise to χ?

(4) Suppose that $\{\mu_n\}$ is a sequence of measures on S^1 and that $\mu_n \to \mu = d\theta$ (Lebesgue measure) in the weak* topology. Does it follow that $\mu_n(E) \to \mu(E)$ for any Borel set $E \subset S^1$? If yes, prove it. If not, find a counterexample.

Part III. Do **three** of the following four problems.

(1) How many zeroes (counting multiplicity) does $\sin(z) + 2iz^2$ have inside the rectangle

$$\left\{z : |\text{Re}(z)| < \frac{\pi}{2}, |\text{Im}(z)| \leq 1\right\}?$$

(2) Let f be a meromorphic function on the complex plane $\mathbb{C}$. Suppose that for every polynomial $p(z)$ and every closed contour Γ avoiding the poles of f we have

$$\int_\Gamma p(z) f(z)^2 \, dz = 0.$$

Prove that f is entire.

(3) Find explicitly a Riemann map (that is, a biholomophic bijection) of the unit disk $\mathbb{D}$ onto each of the following domains:

(a) the extended plane $\hat{\mathbb{C}}$ minus the segment $[-1, 1]$;

(b) the strip $0 < \text{Im}(z) < 1$;

(c) the first quadrant;

(d) the intersection of the unit disk with the upper half plane.

(4) The Fourier transform of a function $f \colon \mathbb{R} \to \mathbb{R}$ is defined by

$$\hat{f}(z) = \int_{-\infty}^{\infty} f(x) e^{-ixz} dx.$$

Does there exist a function $f \in C_c(\mathbb{R})$ (continuous with compact support) such that $\hat{f} \in C_c(\mathbb{R})$? (Hint: Use complex analysis. You do not need any prior knowledge of the Fourier transform).