Preliminary Exam in Algebra Spring 2020

Instructions:

(1) There are **three** parts to this exam: I (Groups, Rings, modules), II (Fields, Galois Theory, and representation theory), and III (Linear and homological algebra). There are **five** problems in each part. You should present good solution to **three** problems from each part; if you present solutions to more than three problems in a part, the grader will select which three solutions contribute most to the total grade.

(2) In each problem, full credit requires proving that your answer is correct. You may quote and use theorems and formulas. If a problem asks you to state or prove a theorem or a formula, you need to provide the full details. If it asks you to disprove a statement, a counterexample will suffice, again of course with full details.

Part I. Groups, rings, modules

(1) Let G be a finite group of order p^k, where p is a prime number.

(a) Show that the center of G contains an element which is not the identity element of G. One possible approach is to consider the action of G on itself by conjugation.

(b) Show that that there is an increasing sequence of subgroups

$$\{e\} = G_0 \subseteq G_1 \subseteq G_2 \subseteq \cdots \subseteq G_k = G$$

with G_i normal in G_{i+1} and G_{i+1}/G_i of order p.

(2) Let S be a multiplicative system in a unital commutative ring A. Let A_S be the localization of A by S. Show that the morphism $A \to A_S$ is onto if and only if every element of S is invertible in A.

(3) Let V be a finite dimensional vector space over the real numbers $\mathbb{R}$ and let $T : V \to V$ be a linear transformation so that $T^n = 1$ for some **odd** integer n. Suppose $T(v) \neq v$ for all $0 \neq v \in V$. Show that V can be written as a direct sum of 2-dimensional irreducible invariant subspaces.

Recall that a subspace W is invariant if $T(W) \subseteq W$ and that it is irreducible if it contains no non-trivial invariant subspaces.

(4) Suppose G is a finite simple group. Determine the number of isomorphism classes of 1-dimensional representations of G. (Your answer may depend on G.)

(5) Let $\mathbb{C}[x, y, z]$ be the polynomial ring in three variables over the complex numbers.

(a) Show that $I = (x, y)$ is a prime ideal and $J = (x, y, z)$ is a maximal ideal of $\mathbb{C}[x, y, z]$.

(b) What are the minimal prime ideals of $(xy, xz, yz) \subseteq \mathbb{C}[x, y, z]$?

Part II. Fields, Galois theory, and representation theory

(1) Compute the Galois group of the splitting field of $x^4 - 2$ over

(a) $\mathbb{Q}(\sqrt{2})$;

(b) $\mathbb{F}_5$.

(2) Suppose K/k is a finite algebraic extension. Let K_s denote the set of all elements of K that are separable over k; this is known to be a subfield of K (you do not need to prove this). Show that

$$[K_s : k] = |\operatorname{Hom}_k(K, \overline{k})|,$$

where $\overline{k}$ is any algebraic closure of k.

(3) Suppose K and L are extension fields of k.

(a) If $\operatorname{Hom}_k(K, L) \neq \emptyset$, prove that $\operatorname{tr.deg} K/k \leq \operatorname{tr.deg} L/k$.

(b) Determine if the converse is true. Give a proof or a counterexample.

(4) Suppose G is a finite group of order 24, and three rows of its character table are given by

V_1	1	-1	1	-1	1
V_2	2	0	-1	0	2
V_3	3	1	0	-1	-1

Let W denote the remaining nontrivial irreducible representation of G. Determine the decomposition of $W \otimes V_2$ into irreducibles.

(5) Suppose G is a finite group. Prove that the number of irreducible complex representations of G whose characters are <u>real-valued</u> is the same as the number of those conjugacy classes of G that are closed under inversion.

Part III. Linear and homological algebra

(1) Let A be an integral domain and K its field of fractions (in other words, its localization A_S where $S = A - \{0\}$).

(a) Give an example when K is not a projective A-module.

(b) Give an example when K is an injective A-module, other than A being a field.

(2) Let k be a field and V a finite dimensional k-vector space. Let $T : V \to V$ be a linear transformation.

(a) Define the minimal polynomial of T and show it exists and is unique up to multiplication by a non-zero element of k.

(b) Prove that T is diagonalizable if and only if the minimal polynomial factors completely over k and has distinct roots.

(3) Let $\mathbb{Q}[x]$ be the polynomial ring over the rational numbers. Define two $\mathbb{Q}[x]$-modules

$$M_0 = \mathbb{Q}[x]/(x) \qquad \text{and} \qquad M_1 = \mathbb{Q}[x]/(x-1)^2.$$

Calculate

$$\mathrm{Tor}_n^{\mathbb{Q}[x]}(M_0, M_0 \oplus M_1), \quad n \geq 0.$$

(4) Let R be a commutative ring and let M and N be two R-modules.

(a) If R is a principal ideal domain prove that for all $n > 1$

$$\mathrm{Tor}_n^R(M, N) = 0.$$

(b) Give an example of R, M, and N so that $\mathrm{Tor}_2^R(M, N) \neq 0$.

(5) Show that a morphism in an Abelian category is a monomorphism if and only if its kernel is zero.