PRELIMINARY EXAM IN ALGEBRA FALL 2022

INSTRUCTIONS:

(1) This exam has **three** parts. Do **three** problems from each part. If you attempt more than three problems in one part, then the three problems with highest scores will count.

(2) In each problem, full credit requires proving that your answer is correct. You may quote and use theorems and formulas. But if a problem asks you to state or prove a theorem or a formula, you need to provide the full details.

Part I. Group Theory, Homological Algebra, and PIDs

Do **three** of the following five problems.

(1) Classify, up to isomorphism, all groups of order 35.

(2) Let p be a prime number, $\mathbb{F}_p$ the field with p elements, and $\mathrm{GL}_n(\mathbb{F}_p)$ the group of invertible $n \times n$ matrices with coefficients in $\mathbb{F}_p$.
   (a) Calculate the order of $\mathrm{GL}_n(\mathbb{F}_p)$. You could use induction on n and that $\mathrm{GL}_n(\mathbb{F}_p)$ acts transitively on the non-zero vectors of $\mathbb{F}_p^n$.
   (b) Write down a p-Sylow subgroup of $\mathrm{GL}_n(\mathbb{F}_p)$ and prove it has the correct order.

(3) Let $\mathbb{Z}[C_3]$ be the group ring over the integers of the cyclic group $C_3 = \{1, t, t^2\}$ with $t^3 = 1$.
   (a) Let C_3 act on $\mathbb{Z}$ by $t(n) = n$. Write down a projective resolution of $\mathbb{Z}$ as a $\mathbb{Z}[C_3]$-module.
   (b) Calculate $\mathrm{Ext}^n_{\mathbb{Z}[C_3]}(\mathbb{Z}, \mathbb{Z})$, $n \geq 0$.

(4) Let k be an algebraically closed field and $T : V \to V$ a linear transformation from a finite dimensional k-vector space to itself.
   (a) Define the minimal polynomial $m_T(x)$ of T.
   (b) Prove that if $m_T(x)$ has distinct roots, then V has a basis of eigenvectors for T.

(5) Let A be principal ideal domain and $S \subset A$ a multiplicatively closed subset. Assume $1 \in S$ and $0 \notin S$. Prove the localization $S^{-1}A$ is also a principal ideal domain.

Part II. Galois Theory and Representation Theory

Do **three** of the following five problems.

(1) Find the splitting field of $x^5 - 2$ over $\mathbb{Q}$ and compute its Galois group.

(2) Let G be a finite group and let G' be its commutator subgroup. Prove that $[G : G']$ is equal to the number of 1-dimensional complex characters of G.

(3) Let $\mathbb{Q} \subseteq K \subseteq \mathbb{C}$ be an intermediate field such that $K/\mathbb{Q}$ is a finite normal extension of odd degree. Prove that $K \subseteq \mathbb{R}$.

(4) Let F be a field, let $f(x) \in F[x]$ be an irreducible polynomial of degree n, let K be the splitting field of f over F, and let $\alpha \in K$ be a root of f. Assume that $\mathrm{Gal}(K/F) \cong S_n$. Prove that $\mathrm{Gal}(K/F(\alpha)) \cong S_{n-1}$.

(5) Recall that doubling the cube means to construct a segment of length $\sqrt[3]{2}$, given a segment of length 1, using only a straightedge and a compass. More generally, doubling the d-dimensional hypercube means constructing a segment of length $\sqrt[d]{2}$, given a segment of length 1, using only a straightedge and a compass. Find the set of all natural numbers d for which doubling the d-dimensional hypercube is solvable.

Part III. Commutative Algebra and Category Theory

Do **three** of the following five problems.

(1) Let $F : \mathcal{C} \to \mathcal{D}$ be a left adjoint functor. Prove that F preserves colimits.

(2) (a) Prove Baer's Criterion: For R a commutative ring, an R-module I is injective if for every ideal $J \subset R$ and every R-module map $f : J \to I$ there exists an extension $\overline{f} : R \to I$.
(b) Prove that $\mathbb{Q}/\mathbb{Z}$ is an injective abelian group.

(3) Prove the following form of Hensel's Lemma. Let R be a Noetherian ring, $I \subset R$ an ideal, and $f(x) \in R[x]$ a polynomial. If the equation $f(x) = 0$ has a solution $a \in R/I$ and $f'(a)$ is a unit in R/I, then there exists a unique lift of a to an element $\overline{a}$ in the completion $\widehat{R}_I$ such that $f(\overline{a}) = 0$.

(4) Let R be an Artin ring. What is the Krull dimension of R? Prove your answer.

(5) Consider the ring $R = \mathbb{C}[x, y]/(y^2 - x^3 + x)$.
(a) Compute the Krull dimension of R.
(b) Construct an inclusion of a polynomial ring A over $\mathbb{C}$ such that the map $A \to R$ is finite.