ALGEBRA PRELIMINARY EXAM, AUGUST 2021

Solve **three** problems from each part below. Full credit requires proving that your answer is correct. You may quote theorems and formulas from the lectures, unless a problem specifically asks you to justify such.

1. Part 1: Groups, Rings and Modules

(1) Let $p < q$ be prime numbers.
  (a) Show that if $p \nmid (q^2 - 1)$ then any group of order pq^2 is abelian.
  (b) Construct a non-abelian group of order pq^2 when $p \mid (q^2 - 1)$.

(2) Determine the isomorphism class of the 2-Sylow subgroup of S_5. Compute the number of 2-Sylow subgroups of S_5.

(3) Consider the ring $R = \mathbb{C}[t^2, t^3] \subset \mathbb{C}[t]$.
  (a) Show that R is not a principal ideal domain.
  (b) Show that R is not a unique factorization domain.

(4) Consider the subring

$$\mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} \mid a, b \in \mathbb{Z}\}$$

  (a) Define the norm $N(a + b\sqrt{2}) = a^2 - 2b^2$. Show that the norm is multiplicative:

$$N(zw) = N(z)N(w)$$

  (b) Show that $z \in \mathbb{Z}[\sqrt{2}]$ is a unit iff $N(z) = \pm 1$.
  (c) Show that the unit group $\mathbb{Z}[\sqrt{2}]^\times$ is infinite.

(5) Let A, B, C be modules over a PID R such that $A \times B \cong A \times C$.
  (a) Suppose A, B, C are finitely generated. Show that $B \cong C$.
  (b) Given an example of the above situation where B and C are not isomorphic.

2. Part 2: Linear algebra and Galois theory

(1) (a) Show that any element of finite order in $\mathrm{GL}_n(\mathbb{C})$ is diagonalizable.
  (b) Show that any non-diagonalizable element of $\mathrm{GL}_n(\overline{\mathbb{F}}_p)$ has finite order m, where $m = m'p^r$ with $p \nmid m'$ and $0 < r < n$.

(2) Find a complete set of representatives for the similarity classes of a matrix $A \in M_4(k)$ which satisfies the equation

$$A^3 - A^2 + 2A - 2 = 0$$

when
  (a) $k = \mathbb{C}$.
  (b) $k = \mathbb{Q}$.

(3) Let K be a field of characteristic p and let L/K be a Galois extension with Galois group $\mathbb{Z}/p$.

   (a) Let σ be a generator for $\mathrm{Gal}(L/K)$. Show that there is an element $\alpha \in L$ such that $\sigma(\alpha) = \alpha + 1$. (Hint: think of $\sigma - 1$ as a nilpotent K-linear endomorphism of the n-dimensional K-vector space L)
   (b) Show that L is the splitting field over K of a polynomial of the form $x^p - x - a$ with $a \in K$.

(4) Let p be an odd prime and let ζ_p be a non-trivial p-th root of unity (in $\mathbb{C}$).
   (a) Show that the extension $\mathbb{Q}(\zeta_p)/\mathbb{Q}$ has a unique quadratic subextension. Find a generator for this subextension. (You may use standard facts about Galois groups of cyclotomic extensions).
   (b) When $p = 7$, show that this quadratic subextension is $\mathbb{Q}(\sqrt{-7})$.

(5) Compute the Galois group of the polynomial $x^4 + 3x^2 + 4$ over $\mathbb{Q}$ and over $\mathbb{F}_3$.

3. Part 3: Homological algebra, commutative algebra and representation theory

(1) For a field K, is $K[t]$ a flat module over $K[t^2, t^3]$? Justify your answer.

(2) For every integer m, view $\mathbb{Z}/m\mathbb{Z}$ as a module over $\mathbb{Z}[x]$ on which x acts by zero. For any two integers m and n compute $\mathrm{Tor}_\bullet^{\mathbb{Z}[x]}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$ and $\mathrm{Ext}^\bullet_{\mathbb{Z}[x]}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$.

(3) Let H be the group $\{\pm 1, \pm i, \pm j, \pm k\}$ subject to
$$i^2 = j^2 = k^2 = -1;\ ij = -ji = k;\ jk = -kj = i;\ ki = -ik = j.$$
Let σ be the automorphism of H such that $\sigma(-1) = -1$ and
$$\sigma : i \mapsto j \mapsto k \mapsto i.$$
Let G be the group generated by its subgroup H and by an element γ of order 3 subject to relations
$$\gamma h \gamma^{-1} = \sigma(h)$$
for $h \in H$.
   (a) List the dimensions of irreducible representations of G over $\mathbb{C}$.
   (b) Prove that G is not isomorphic to S_4.

(4) Let A be a PID and let a be a non-zero element of A. Show that $A/(a)$ is an injective module over itself.
Is the statement true for $A = k[t^2, t^3]$?

(5) Prove the following formulation of Nakayama's lemma: Let I be an ideal of a commutative unital ring A. Let M be a finitely generated A-module such that $IM = M$. Prove that there exists an element x of I such that $(1 + x)M = 0$. Find a counterexample when M is not finitely generated.