ALGEBRA PRELIMINARY EXAM, SEPTEMBER 2019

INSTRUCTIONS: Do **three** problems from each part below. In each problem, full credit requires proving that your answer is correct. You may quote and use theorems and formulas, but if a problem asks you to state or prove one such, you need to provide the full details.

Part 1: Groups, rings and modules

(1) Prove that there are no simple groups of order 30.

(2) Let G be a finite group of order m, and let H be a normal subgroup of G of prime order p. Assume that m and $p-1$ are coprime.
(a) Compute the order of $\mathrm{Aut}_{\mathrm{Grp}}(H)$.
(b) Show that H is contained in the center of G.

(3) Show that $\mathbb{Z}[X]$ is not a PID. Deduce that a subring of a PID is not necessarily a PID.

(4) Consider the subring of $\mathbb{C}$:
$$\mathbb{Z}[\sqrt{-5}] = \{a + bi\sqrt{5} \mid a, b \in \mathbb{Z}\}.$$
(a) Define a norm N on $\mathbb{Z}[\sqrt{-5}]$ by setting $N(a + bi\sqrt{5}) = a^2 + 5b^2$. Prove that $N(zw) = N(z) \cdot N(w)$, and deduce that the units in $\mathbb{Z}[\sqrt{-5}]$ are 1 and -1.
(b) Prove that 2, 3, $1+i\sqrt{5}$, $1-i\sqrt{5}$ are all irreducible nonassociate elements of $\mathbb{Z}[\sqrt{-5}]$.
(c) Prove that $\mathbb{Z}[\sqrt{-5}]$ is not a UFD.

(5) Let R be an integral domain, and recall that the dual of an R-module M is $M^\vee = \mathrm{Hom}_R(M, R)$.
(a) Show that $M^\vee$ is a torsion-free module for any M.
(b) Define a canonical map $\varphi \colon M \to M^{\vee\vee}$. Show that if this map is injective, then M is torsion-free.
(c) Give an example of a module M over an integral domain such that the map φ is not injective.

Part 2: Fields, Galois theory and representation theory

(1) Let k be a field, $f \in k[X]$ a monic irreducible polynomial of degree n, and K a splitting field of f.
(a) Show that $[K : k]$ divides $n!$.
(b) If $k = \mathbb{F}_p$, show that $[K : k] \leq n$.

(2) Let $\zeta_8 = e^{2\pi i/8}$ be a primitive root of unity of order 8, and consider the cyclotomic extension $\mathbb{Q} \subset \mathbb{Q}(\zeta_8)$. Provide details for the following questions:
(a) Compute the minimal polynomial of ζ_8 over $\mathbb{Q}$.

(b) Compute the Galois group of the extension.
(c) How many intermediate extensions (i.e different from $\mathbb{Q}$ and $\mathbb{Q}(\zeta_8)$) are there?

(3) Determine the Galois group of the polynomial $X^3 - 2$ over $\mathbb{Q}$, and describe explicitly all the subfields of the splitting field of this polynomial.
(4) Find the character table and all the irreducible representations of S_3; include details. Write down the decomposition of the regular representation of S_3 into irreducible representations.
(5) Let G be a finite group, Z its center, and V an irreducible complex representation of G.
(a) Show that every element $z \in Z$ acts on V by multiplication by a scalar.
(b) Show that $(\dim V)^2 \leq \frac{|G|}{|Z|}$.

Part 3: Linear and homological algebra

(1) Let k be a field. A matrix $A \in M_n(k)$ is nilpotent if $A^k = 0$ for some integer k.
(1) Find all possible Jordan canonical forms of a nilpotent matrix A when $n = 5$.
(2) Show that if $A^k = 0$ for some integer k, then $A^k = 0$ for some integer $k \leq n$.
(3) Show that the trace of a nilpotent matrix is 0.

(2) Prove the following version of the "four lemma": if

$$\begin{tikzcd} M_1 \arrow[r] \arrow[d, "\alpha"] & M_2 \arrow[r] \arrow[d, "\beta"] & M_3 \arrow[r] \arrow[d, "\gamma"] & M_4 \arrow[d, "\delta"] \\ N_1 \arrow[r] & N_2 \arrow[r] & N_3 \arrow[r] & N_4 \end{tikzcd}$$

is a commutative diagram of R-modules with exact rows, α and γ are epimorphisms δ is a monomorphism, then β is an epimorphism.

(3) Let G be a finitely generated abelian group of rank r. Show that:
(a) $G \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \mathbb{Q}^r$.
(b) For infinitely many primes p, $G \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z} \simeq (\mathbb{Z}/p\mathbb{Z})^r$.

(4) Let k be a field, and n and m positive integers. Compute

$$\mathrm{Tor}_i^{k[X]}\left(k[X] \oplus \frac{k[X]}{(X^n)}, k[X] \oplus \frac{k[X]}{(X^m)}\right)$$

for all $i \geq 0$.

(5) Prove that no nontrivial $\mathbb{Z}$-module is both projective and injective.